DISCOVERY IN MATHEMATICS

A TEXT FOR TEACHERS

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PREFACE

The Madison Project, led by Dr. Robert Davis, was most visible in the 60's when these materials were first published, and when the National Science Foundation assembled a team of mathematicians and teachers of mathematics to train large numbers of elementary school teachers in Chicago, St. Louis, San Diego, Los Angeles, New York, Philadelphia, and other major cities. The teachers were excited by these new discovery materials, by the approaches which made algebraic topics accessible to young children, and by the possibility of opening new frontiers of mathematics to boys and girls who heretofore were usually confined to exercises in computation and simple word problems from text books.

The National Science Foundation team also included educators from Europe. These people, with the other imaginative teachers on the team, introduced new topics and alternative approaches, including the use of physical materials like Cuisenaire® rods, Dienes Multibase Arithmetic Blocks, geoboards, and the units developed by Elementary Science Study. The Madison Project was further influenced by Dr. Davis' studies of the ways in which children learn.

It is rare indeed that a book grows as dramatically as *Discovery* has grown since the first printing. The fine topics of "Classical Madison Project" are here once again, tested by time, and complemented by new approaches and new materials introduced by the many fine, unselfish people identified with the Project. The book has grown because the people of the Project have grown, and have learned much from each other and from their leader, Bob Davis.

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NOTE: The preceding chapters carry the study of algebra to a quite advanced level. In the following chapters, the study of geometry is begun at a very simple level. Consequently, Chapters 42 through 50 are much easier than Chapters 30 through 41.

ASSESSING THE PROBLEM

Are School Math Programs Getting Better?

In recent years we have been hearing quite a few reports of the successes and failures of school mathematics. The question *is* important, because several studies indicate that many adults find their education and careers blocked by weaknesses in their knowledge of mathematics. Weak mathematical backgrounds, for example, are a major obstacle to the admission of women into medical schools, and of blacks into engineering school [c.f., e.g., Ernest, 1976; Sells, 1973]. There is abundant evidence that mathematically gifted students are often neglected, so that their interest is not aroused, and their potentially great abilities are not developed. For all kinds of students an insufficient mastery of mathematics can be severely limiting. In today's world we know that *anyone* who misses out on learning mathematics has lost out on something very valuable.

How well are school math programs serving today's students? Most of the reports that one reads indicate that achievement scores on nation-wide tests reveal *declines*—things are getting worse. This is especially true in regard to those test items that deal with problem-solving, or with the creative use of mathematics—in short, *with the ability to use mathematics in almost any situation that one really cares about*. [NAEP, 1970]

This discouraging news has occupied the headlines, partially obscuring the fact that *some* school programs have been *improving*. These programs are doing better than ever by their students, in the sense that the students show important learning gains *both in computational skill and also in conceptual understanding. These students are learning more mathematics than ever before!* [Dilworth, 1975; Conference Board of The Mathematical Sciences, 1975]

Clearly, we need to pay special attention to those schools that are showing substantial improvements in student learning. While nation-wide testing programs do indicate some declines in achievement for the "average" United States school *there are some schools showing improvement.* How do these "successful" schools do it?

One important observation about these successful programs is that they deal effectively with *three key aspects of mathematics*:

(i) the *computational skills* of mathematics;
(ii) the *ideas* of mathematics;
(iii) the *uses* of mathematics.

Indeed, no student can truly be said to have learned mathematics unless he has become skillful in dealing with all three of these aspects, separately and (more importantly) in combination. *In the real world, it is the combination of all three of these ingredients that is usually important.*

The question then becomes this: how do the teachers in these successful classrooms manage to create learning experiences for their students that combine all three ingredients—skills, ideas, and uses—into an enjoyable and valuable whole? *It is the purpose of this book to provide part of the answer to this question.* This book is not the whole story—for the complete program, one must look to a teacher who is combining the ingredients with the skill of a master chef or an inspired composer. But this book can provide some of the essential ingredients.

Parents Are Important, Too

I have spoken of "teachers" providing for the educational needs of their students, because that is what we have usually observed. But the demands on schools and teachers are almost unbearable, and often seem to be getting worse. It is not always possible for teachers to provide everything that one would like the students to experience. Parents can play a major role. (With some of my own best students, I suspect that the children learn more from their parents than they do from me. No matter—it's still a great joy to see how well the students progress toward a stronger and stronger command of mathematics!*)

Before turning to the important question of *how* one builds a strong "3-ingredient" math program, it will be wise to make sure that we agree on what the three ingredients actually are. We do this in the following section.

A Strong Mathematics Program Has Three Parts

We have said, very briefly, that "a strong education in mathematics must provide for three ingredients—*skill* in calculation, good understanding of the *ideas* of mathematics, and a comfortable facility in *using* mathematics in many different situations." We now explain these ingredients in somewhat greater detail.

Ingredient Number 1.—Competence in Computational Skills

Students need to learn that 7 + 4 = 11, how to multiply 1066 by 340, and how to solve word problems. There are many textbooks series that deal well with these topics (for example, Denholm, Hankins, et al., 1980), and I assume that your school already uses one such series, so the present book does not deal much with this aspect of school mathematics. *I assume that you will continue to use a good standard arithmetic textbook, and to provide a generous amount of practice in calculation.* (Research does show, however, that adding the two other ingredients to your school program will tend to *improve* student performance on calculation. Cf., e.g., Hopkins, 1965. So you need not lose out on computation because you add "concepts" and "applications"—on the contrary, the Hopkins study indicates that your students' computational proficiency can also be improved!

Ingredient Number 2. The Ideas of Mathematics

Considerable evidence shows that computational skills, alone, are not enough. Mathematics involves also *ideas.* This is one of the places where many school mathematics programs fall down: they do teach skills, but they neglect the key *ideas* of mathematics.

What are the main ideas of beginning mathematics? They include the idea of a mathematical *variable*, of a mathematical *function*, of a *graph*, and so on. In later pages of the book we will explore these ideas carefully. But, to get an initial broad overview, let me divide the key ideas of elementary mathematics into those ideas that deal mainly with arithmetical operations and problem solving—I shall call this strand *algebra*, although the various parts of mathematics are not really sharply delineated and they tend to over-lap quite a bit—vs. a second category that includes those ideas that deal with shapes, positions, directions, and motions. This latter strand I will call *geometry*. What is meant in each case will be explained further below.

Ingredient Number 3. The Uses of Mathematics

Every advancing human society has developed mathematics. Probably the main reason has been that mathematics is *useful*, though a secondary reason has been that mathematics is often quite interesting. We will not try to separate these two reasons, because that is often almost impossible to do. Did we, for example, send astronauts to the moon because it was *interesting* to do so, or because it was *useful* to do so?

A strong school mathematics program must

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*To see some examples of student work, refer to *The Journal of Children's Mathematical Behavior*, vol. 2, no. 2 (1979).

Then a man came in and bought an ant farm for $5. (At this point was there *more* money in the cash register than when we opened up this morning, or was there *less*? Neither! There is just the same amount as when we unlocked the door this morning: $15 - 20 + 5 = 0$.)

(32) $15 - 20 = ?$	(32) **$^{-}5$**
(33) $15 - 20 + 3 = ?$	(33) **$^{-}2$**
(34) $10 - 6 = ?$	(34) **$^{+}4$, or 4 (no sign, as in 4, means the same thing as a positive sign, as in $^{+}4$).**
(35) $6 - 10 = ?$	(35) **$^{-}4$**
(36) $6 - 10 + 8 = ?$	(36) **$^{+}4$**
[page 15]	
(37) $6 - 10 + 4 = ?$	(37) **0**
(38) $6 - 10 + 3 = ?$	(38) **$^{-}1$**
What answers do you get for these problems?	
(39) $^{+}5 + ^{+}3 = ?$	(39) **$^{+}8$**
(40) $^{+}5 + ^{-}2 = ?$	(40) **$^{+}3$**
(41) $^{+}10 + ^{-}1 = ?$	(41) **$^{+}9$**
(42) $^{-}6 + ^{-}1 = ?$	(42) **$^{-}7$**
	For problems 39 through 42, it is advisable not to ask the children to make pet store stories. It probably will not be necessary, and it might be confusing. If you *do* want story interpretations for problems 39 through 42, you can use the matrix game, or simply say: A *gain* of $5 and a *gain* of $3 mean . . . or positive five plus positive three equals positive eight. The "pet shop" model really permits us to add and subtract unsigned numbers, and to represent the result as a signed number. The "postman" model (presented below) is necessary for a discussion of adding and subtracting signed numbers.
Can you find the truth sets for these open sentences?	
(43) $^{+}8 + \square = ^{+}6$	(43) **$\{^{-}2\}$** If a child answers "positive two," ask the children, "Positive eight plus positive two equals what?"
(44) $^{+}8 + \square = ^{+}8$	(44) **$\{0\}$** If children say $^{+}0$, or $^{-}0$, accept either, usually without comment, unless you feel the children are really ready to observe that $^{+}0 = ^{-}0 = 0$, and even then it is better to wait until the comment comes from them. As an alternative, you can precipitate their discovery by asking, in a puzzled tone of voice, "Which should it really be, $^{+}0$ or $^{-}0$?" If you sound as if *you* don't know, one of the children is almost sure to explain it to you.
(45) $^{+}12 + \square = 0$	(45) **$\{^{-}12\}$**
(46) $^{+}5 + \square = ^{-}5$	(46) **$\{^{-}10\}$**
(47) $^{+}105 + \square = ^{+}101$	(47) **$\{^{-}4\}$**
(48) $10 + \square = 17$	(48) **$\{7\}$ or $\{^{+}7\}$**

show children how mathematics is *useful*, how it relates to things in the real world, and why this can often be extremely interesting. This, unfortunately, is also a place where many school programs are weak, and where considerable improvement is often needed.

All Three Ingredients Must Be Brought Together

The heading of this section tells the whole story. Each of the three ingredients listed above is necessary, and all three ingredients must fit together. *Each of them is strengthened by the presence of the other two.* A child who has been neglecting his study of arithmetic may turn to it more seriously if he becomes interested in using mathematics to study waiting times in the school cafeteria, or to figure out how much it would cost to get and keep a pet. A child who has not yet understood the size of the number "2/3" may get a far clearer idea as a result of using Cuisenaire® rods. A child who calculates incorrectly in subtraction problems such as:

$$\begin{array}{r} 7{,}003 \\ \underline{-\ 295} \end{array}$$

may see the error, and correct it, as a result of using play money to represent "ones," "tens," "hundreds," and "thousands." There is virtually no end to the number of examples that can be given. In Cantonese cooking, by putting the ingredients together correctly one can make a magnificent sweet and sour pork—*the whole becomes greater than merely the sum of the parts.* The same thing happens with the combining of notes and themes and counterpoint to compose a great piece of music. And the same thing happens with a strong school mathematics program—the ingredients, properly blended together, are much more effective than any of them (or even *all* of them) taken separately.

This Book Deals Mainly With Only One Ingredient

The present book deals primarily with the *ideas* of mathematics. We leave it up to each teacher to make his or her own sweet and sour pork, or great symphony (or whatever), by combining all three ingredients in a skillful and artistic way. We urge one thing: *remember that all three ingredients—skill, ideas, and uses—must be brought together in order to get a strong program in school mathematics.* The present book comments briefly on some interesting uses of mathematics, and then proceeds with a careful development of some of the key *ideas.*

What Grade Level Are We Talking About?

The teachers who have developed these materials—in Weston, Connecticut, in Webster Groves, Missouri, in San Diego, California, in Syracuse, New York, in Urbana, Illinois, and elsewhere—work in different school situations, and have used the materials appropriately *for their situations.* Two main patterns emerge:

1) The use of these materials to provide a complete "three-ingredient" mathematics program in grades 3, 4, 5, and 6; or else
2) The use of these materials to provide a complete "three-ingredient" mathematics program in grades 7, 8, and 9.

Both of these patterns seem to work successfully, and there are surely many more modifications that could be made, in order to adapt to local situations.

The Madison Project Is People

The mathematicians, teachers, and administrators who created the school mathematics program described (in part) in this book refer to their joint venture as "The Madison Project,"because it was first attempted in The Madison School in Syracuse, N.Y., beginning in 1957. As other teachers have observed classes, become interested, and come to the Project to study methods of teaching a "3-ingredient" program, the number of teachers who are expert users of this approach has grown quite substantially. This large group of experienced, expert teachers is important to everyone considering the use of these materials. *If you, as a teacher, want help in teaching a "3-ingredient" program there may be a teacher reasonably near you*

who could offer that help. Of course, it works both ways: if you are teaching a "3-ingredient" program of this type successfully, then let us know about it, especially if you would be willing to help other teachers in your area. (The same offer—and request—applies to parents, too.)

Over two decades of experience have convinced us that we can all improve our curriculum and our teaching, and that the best way to do it is in cooperation with other excellent and experienced teachers. That such arrangements are often possible with regard to the Madison Project's "3-ingredient" program is one of its greatest strengths. For information on teachers in your area, please write to us. (Also, to volunteer your own services to help others.)

II. TYPES OF LESSONS

Observation of different classrooms quickly reveals a variety of styles of teaching, and a variety of classroom activities used by different teachers. The teachers who developed the present materials generally use a deliberate diversity of lesson types that can be distinguished on three dimensions: first, the lesson may make use of physical apparatus or physical materials (such as Cuisenaire® rods, geoboards, protractors, etc.), or it may not. Second, lessons can be classified by pedagogical *purpose*, as either a *discovery lesson*, or as an *exploratory lesson*, or as an *experience lesson*, or as a *practice lesson*, or as a *mastery lesson*, or as a *challenge lesson*. Finally, lessons can be classified by the kind of classroom organization that is employed: a whole class organization, or a small group organization, or individual work.

We can illustrate some of these types by reference to a lesson that is used to introduce the concepts of *average* (or "mean") and *variance*. As taught by some teachers, a class of (about) 30 students might be divided into 10 teams of 3 each. The initial task is to determine the length of the classroom. First, each team prepares their best *guess*. This would be called a "small-group" organization. When the guesses have been decided upon, the class switches to a *whole class* organization: the 10 guesses are written on the front board, their average is calculated, and their variance* is computed, with everyone in the class (hopefully!) participating, or at least paying careful attention.

Now the class returns to a small group format, and the 10 groups each *measure* the length of the classroom, using cheap 6-inch plastic rulers. This method is not highly accurate, but may be an improvement on the guesses. (The inaccuracy is a desirable feature, at this stage in the work—we hope to find *more* agreement than in the case of guesses, but we want to leave room for still better agreement in subsequent stages of the work, when yet more accurate methods of measurement are employed.)

When each group has completed its measurements, the class again shifts to a whole class organization, and the average and variance of these measurements are computed, and compared with the results for the guesses, with the entire class involved in this comparison.

This process of switching back and forth between small group and total class organization continues, with the small groups next using good-quality meter sticks (or yardsticks), and in the final cycle using a good quality surveyor's tape measure.

This is an interesting lesson, and—among other things—offers a chance to practice arithmetic in an interesting setting. (Of course, it also introduces the ideas of *average* and *variance*.) I cite it here, however, in the hope of clarifying the meaning of "small-group" classroom organization, vs. "total class" organization. If the teacher (or a student) is standing at the front of the room, and everyone else is watching (or is supposed to be watching), that represents "whole class" organization—the class is attempting to work as a single unified group, with everyone paying attention to the same thing.

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*The *variance* is a number that tells how much agreement (or disagreement) there is among the different guesses. Cf., e.g., Robbins and Van Ryzin, 1975.

By contrast, in "small group" organization, a definite group of three (or four, or five) students are working together; meanwhile, *another* group of 3 (or 4, or 5) students work together; and around the room one sees, in fact, 4, or 5, or even more *separate* small groups, each attempting to work together on some definite task. (Different groups may be working on the same task, or may be working on different tasks.)

Perhaps we need to look more carefully at the different *purposes* of different lessons. It has been our experience that observers who disagree about a lesson are in fact often assuming different *purposes* for that lesson. I propose to distinguish six different kinds of lessons: discovery lessons, exploratory lessons, practice lessons, experience lessons, mastery lessons, and challenge lessons.

Exploratory Lessons

Many people who seem to be good learners show a certain distinctive behavior when you hand them a new gadget or a new puzzle. They "play around with it" — that is to say, they move some of its movable parts (if any) back and forth, observing carefully as they do it. This can seem to be rather purposeless, but a great many good problem solvers go through this stage.

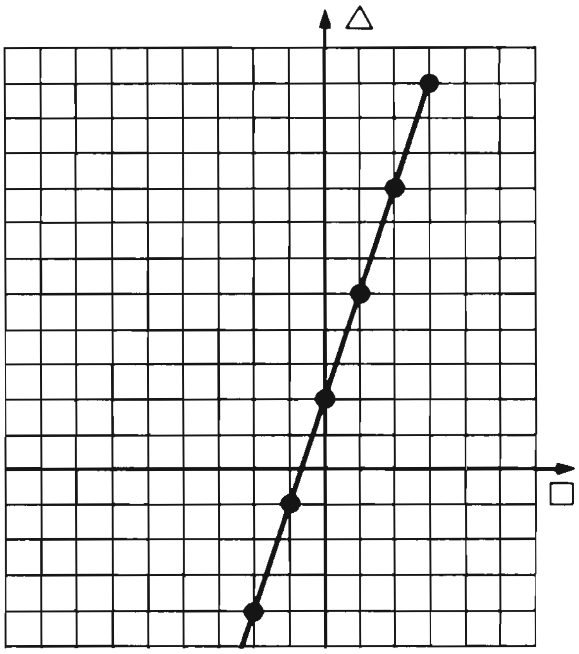
It seems to us that many of our successful students go through this stage, also, so we use "exploratory lessons" to provide opportunities for all students to go through this stage. Suppose, for example, we want to introduce work on graphs. We might begin with an *exploratory lesson*, based on plotting number pairs in truth sets, to let students get some *general, introductory* ideas about how changes in equations correspond to changes in graphs. If this is a true "exploratory lesson," we will want to emphasize diversity — "try something different, and see what happens" — and careful observation — "see if you can see how it works."

Discovery Lessons

This is probably the best known feature of the Madison Project mathematics curriculum. Often — but not always! — when we want to introduce an important new idea, we introduce it by way of a "discovery lesson." For example, there is a very important relation between *this* number in an equation

$$(\square \times \overset{\downarrow}{3}) + 2 =$$

and the pattern that one can see on the corresponding graph.*



If we introduce this by a "discovery lesson," *we will NOT tell the students what the relation is.* (This surprises many observers!). Instead, we work through one example, then another, then another.... Whenever a student discovers the relation, there is one thing she/he does, and one thing she/he does NOT do. The student demonstrates the discovery of "the secret" by using the secret to

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*This pattern will be explained clearly in Chapter 11.

give correct answers, very quickly. That establishes beyond any doubt that the student has discovered the secret—and that it impresses the other students with a very important fact about mathematics—*you can often discover the answer by yourself, even if nobody has told you, if you will really think hard about the problem.*

What the students who have discovered the secret do NOT do, is this: they do NOT *say* what the secret is. *They use it, but they don't tell it.*

Over twenty years of experience convinces us that such "discovery lessons" can be very valuable. But why? In fact, we are not sure. Discovery lessons were developed, not from any abstract theory, but from the experience of many teachers, who found out that such lessons seem to be an important addition to a mathematics program. Many reasons have been suggested [cf. Davis, 1966], including these:

(i) They provide variety (in everything else, the teacher usually *does* tell you);

(ii) they make it clear to students that *they* have the responsibility for observing carefully, and for noticing the key patterns —discovery lessons proclaim to students: "The buck stops here—with you!"

(iii) discovery lessons provide feedback to the teacher; when a teacher lectures, the teacher cannot really tell whether most students are listening, but in a discovery lesson there can be no doubt as to who is participating;

(iv) sometimes it is easier to *show* something to people than it is to *describe* it to them; to *tell* you must describe, but by discovery you can *show*;

(v) in a discovery lesson, students either make the discovery themselves, or else they see their classmates make the discovery—this first-hand participation and observation should prove that mathematics *is discoverable*, that when in doubt you don't have to quit: *by thinking hard about the problem you may be able to discover the answer.*

(vi) there is abundant evidence that many students respond well to a challenge (cf. the case of basketball); when the teacher tells you, it may seem that there is not much challenge—but when you have to figure something out for yourself, the challenge is unmistakable. (In fact, teachers who are skillful at using "discovery lessons" are usually able to adjust the challenge so that, ultimately, *every* student makes the relevant discovery. Methods for doing this are discussed later in this book.)

Practice Lessons

This is perhaps the most familiar type of lesson: when there is a procedure that students really need to be good at, teachers need to provide plenty of practice. (One word of caution, though: it is important not to waste a student's time. Some students do not seem to need much practice, and are able to retain knowledge and skills without requiring much practice. Such students can spend their time more profitably in other kinds of lessons.)

Experience Lessons

"Experience lessons" differ from practice lessons in a subtle but important way. A student practicing long division, or adding fractions, or factoring polynomials, is indeed having a "practice lesson." But consider a student who is looking at triangles drawn on the blackboard, estimating their measure in degrees, and then measuring them with a protractor to see how close his guess was. This has a subtly different quality to it—the student is getting *experience* with an unfamiliar task, not *practicing* a previously learned one.

Mastery Lessons

For the most essential skills and concepts, we want every student to master them, and to get them essentially correct. In such cases one uses *mastery lessons*, which are no-holds-barred feats of tutorial determination. (But not every topic needs to be treated this way!)

Challenge Lessons

What teacher can be sure of finding the proper level for each student? As a precaution, Madison Project teachers use occasional "challenge" topics (or lessons): these are difficult problems which few, if any, students will solve. But if these problems are chosen correctly, some students *will* solve them—and will feel a well-merited sense of accomplishment. After all, who wouldn't want to hit home runs like Reggie Jackson? Good, after all, is *good*.

III. THE ART OF TEACHING

I have seen enough excellent teachers so that I am hesitant to try to tell anyone else how to teach, given the real possibility that they are already better than I am. But there are one or two observations about excellent teaching that may be worth passing on.

1. It often helps if a teacher *accentuates the positive*. If a student's answer is partly right, and partly wrong—which often seems to happen in mathematics—should the teacher respond *mainly* to

what is correct, or mainly to what is wrong? It is often preferable to respond to what is right—if a student added this column of numbers, to get this answer

$$\begin{array}{r}918\\231\\+\,567\\\hline 1706\end{array}$$

he is mostly right! To be sure, he forgot to "carry the one," but look at how many additions and "carrys" he did correctly! (By my count, he did eight operations correctly, so his score is really "8 wins, 1 loss.") There is plenty of reason here for a teacher to emphasize (and praise) what is right (after which one can go on to point out what is wrong). Of course there are exceptions, and a teacher who has a sensitive awareness of a child can know whether on this particular day, to this particular child, it will be better to emphasize the 8 correct operations, or the one incorrect one.

2. Perhaps the biggest criticism of math lessons is that they are so often *dull, boring, uninteresting* [Fey, 1979]. *This is not inevitable.* There is much in mathematics that is fun, or that is interesting, or that is exciting, or that is challenging. One example: my classes (in grades 5 and 6) have nearly always enjoyed a lesson where the class is divided into small groups (or teams) of about 3 students each. We go into the school yard, and each team tries to determine the height of the school's flagpole. I do not tell students how to do it—each team has to make up its own method for solving the problem. We see who comes closest to the correct answer. (And how do we find the "correct" answer? One way is to have the teacher solve the problem beforehand, if the answer is not already known from a previous year's work.) The most common solution uses a protractor to measure the angle of sight as you look through a drinking straw at the top of the flagpole, but there are many other possibilities. The key idea, usually, is to make an accurate scale drawing. (We discuss this lesson further below, in Section 10 of this introduction.)

IV. MEANING

Perhaps no aspect of teaching or learning mathematics is more important than this one. *There are two different ways to learn mathematics: one, so that the symbols have clear meanings, and the other, so that the symbols are meaningless.*

Let me give an example. Suppose we have the problem

$$\begin{array}{r}43\\+\,42\\\hline\end{array}$$

We can, if we choose, teach the addition algorithm in this case by telling children to deal with each column separately:

$$\begin{array}{rr|c|}\cline{3-3} & 4 & 3 \\ + & 4 & 2 \\ \hline & 8 & 5 \\ \cline{3-3} \end{array} \quad \text{and} \quad \begin{array}{r|c|r}\cline{2-2} & 4 & 3 \\ + & 4 & 2 \\ \hline & 8 & 5 \\ \cline{2-2} \end{array}$$

$$3 + 2 = 5 \qquad 4 + 4 = 8$$

so that our complete, final answer is

$$\begin{array}{r} 43 \\ +42 \\ \hline 85 \end{array}$$

I would classify this as a "meaningless" way of teaching the addition algorithm. We have told the students what to do—deal with the two columns separately and independently—*but we have given them no "reasons" why this is appropriate.*

Observations suggest that, in the United States today, most mathematics in early school years is taught in this "meaningless" way. (In some cases this may actually be necessary, because the "meanings" may be so involved that most children would tend to lose interest before they arrived at the point of understanding.)[1]

However, for young children, it is nearly always better to teach mathematics so that the symbols DO have meanings.

For the problem

$$\begin{array}{r} 43 \\ +42 \\ \hline \end{array}$$

we can do this in many ways. One common way is to use money: if the "3" and the "2" represent one dollar bills, and if the "4" and "4" both represent ten dollar bills, the reason we add the way we do can easily be made clear.

We often say: we'd like every child to be able to think about each mathematical statement as a *story about reality*.

Thus, 3 + 2 = 5 could mean "if I have 3 pennies in my left hand, and two pennies in my right hand, and if I put them all together, I can count up and see that I have 5 pennies."

Notice that *meaning* can be very helpful. In the case of addition, if we have a problem like

$$\begin{array}{r} 28 \\ +53 \\ \hline \end{array}$$

the knowledge children have about making change (trading ten pennies for one dime, or ten one's for one ten dollar bill) can be called upon to justify the procedure of "carrys" from one column to the next.

To pursue this idea of *meaning* a bit further, we turn to a more interesting example.

The symbol

$$2371$$

can easily be given a meaning in terms of play money: the "1" refers to one *dollar bill*; the "7" refers to *seven ten dollar bills*; the "3" refers to three *hundred dollar bills*; the "2" refers to two *thousand dollar bills*. (Instead of money, one might use Dienes' MAB blocks, or Patricia Davidson's "chip trading," etc.)

A problem such as

$$\begin{array}{r} 2371 \\ -\quad 5 \\ \hline \end{array}$$

can then have a meaning. You have the money described above, and you want to give someone five dollars. Well, you cannot immediately do this, since you don't have five one dollar bills. But you can *get* more one dollar bills, by giving one ten dollar bill to a banker, and getting ten one dollar bills in exchange. Proceeding in this way, it is easy for a child to learn a *meaning* for such operations as changing

$$2371 \quad \text{to} \quad 23\overset{6}{\cancel{7}}{}^{1}\,1,$$

for the process of subtracting

$$\begin{array}{r} 23\overset{6}{\cancel{7}}{}^{1}1 \\ -\quad 5 \\ \hline 236\;6 \end{array}$$

and so on. This seems to be a far better way for a young child to learn mathematics.

Unfortunately, this is NOT the way it usually happens. More often, beginning mathematics is taught, and learned, as a ritual of meaningless marks on paper. One result is that students make errors of the following type:

Ann, grade 4: Given the problem

$$\begin{array}{r} 7{,}003 \\ -\;\;594 \\ \hline \end{array}$$

Ann realized that she could not subtract 4 from 3, so she "regrouped" (or "borrowed") like this:

$$\begin{array}{r} \overset{6}{\cancel{7}},00^{1}3 \\ -\;\;59\;4 \\ \hline \end{array}$$

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[1] And, at more advanced levels of study, it is important for successful students to develop skill in dealing with mathematics both ways: *with* "meaning" and "understanding," or else as a *meaningless* process where one carries out a certain procedure "without questioning one's orders," as it were. But for younger students, "math with meaning" is usually better than "math without meaning."

and proceeded to subtract 4 from 13. Notice what Ann has done: in terms of play-money meanings, she has given the banker one one-thousand-dollar-bill and accepted 10 one-dollar-bills in return. Surely not a transaction she would be inclined to accept with money! But, with *meaningless symbols,* Ann was quite content to change

$$7{,}003 \quad \text{to} \quad \overset{6}{\cancel{7}}\,{,}00^{1}3 \text{ (or 6,013).}$$

V. MATHEMATICAL KNOWLEDGE

Reflect, for a moment, on the *kinds* of mathematical knowledge that you have learned. You can probably distinguish several different kinds. This is important enough to deserve some discussion. I want to describe five different *kinds* of mathematical knowledge: visually-moderated sequences, integrated sequences, "frames" (also known as "schemata," or "scripts"), planning knowledge, and heuristics.

Visually-Moderated Sequences

The visually-moderated sequence (or "VMS") is in some ways the most basic kind of mathematical knowledge. It consists of something the student *sees* on paper, such as

$$21\,\overline{)3874}$$

then a *memorized procedure* that the student recalls and uses, such as "2 goes into 3 once, so I write a '1' over the '8,' which leads to a *new or modified visual input*

$$\begin{array}{r} 1 \\ 21\,\overline{)3874} \end{array}$$

which, in turn, serves to remind the student of another piece of *memorized procedure* ("multiply the '1' by the '21' and write it under the '38' ")

$$\begin{array}{r} 1 \\ 21\,\overline{)3874} \\ \underline{21} \end{array}$$

Now, this new visual input leads to another piece of memorized procedure ("Oh, yeah, subtract the '21' from the '38' "). And this piece of procedure leads to a new visual input, namely

$$\begin{array}{r} 1 \\ 21\,\overline{)3874} \\ \underline{21} \\ 17 \end{array}$$

and so on. The sequence continues until, one hopes, the problem is solved.

A sequence of this sort—
...visual stimulus reminds student of a thing to do...
...doing that thing leads to a new visual stimulus...
...new visual stimulus reminds student of thing to do...
...doing *that* thing leads to *new* visual stimulus...
...(and so on)

—is known as a *visually-moderated sequence* [cf. Davis, Jockusch, and McKnight (1978); Davis and McKnight (1979)].

For emphasis, let me give an example of a VMS sequence *outside* of mathematics. Suppose you are driving to your brother's farm, located out in the country in New England. You've driven there once before. You are not sure how to get there.

But you decide to try, anyhow.

What you are planning (and hoping) is something like this: you know you should leave town going north on route 59. So you do that. Now, you hope that, before you are irretrievably lost, you will come to some landmark that you can recognize—"oh, yes, there's that *peculiarly* shaped tree. I know—here I'm supposed to turn right!"—and now you turn right, continue driving, and hope to recognize some landmark that will remind you of what to do next. This, too, is a visually-moderated sequence.

(If you watch students at work, you will often see examples of VMS's. Some algebra students, for example, asked to factor

$$x^2 - 5x + 6 =$$

will sit for awhile, then finally write

$$x^2 - 5x + 6 = (\quad)(\quad).$$

Now they are off and running. The parentheses remind them of the next step to take...and so on.)

Many students—and not a few adults—believe that VMS sequences make up the whole of mathematical knowledge. This is a wrong and harmful view of what it means to "understand" mathematics. A VMS sequence is an *insecure* kind of knowledge. If you forget one little piece of it, somewhere, the whole long sequence may take you in an entirely wrong direction. Furthermore, it is often important not just to *have* ideas, but to *think about* these ideas. Since a VMS sequence is dependent upon inputs from the outside world, it is difficult to *think about* the VMS sequence (for example, it is hard to think about it while you are shaving, jogging, or riding a bicycle).

Fortunately, *given enough practice* a VMS sequence acquires a more self-contained quality, no longer depending upon external inputs. In this new form, it is called an *integrated sequence*.

Integrated Sequences

An *integrated sequence* differs from a VMS in that *the integrated sequence does not depend upon frequent visual inputs.*

For teachers, the long division algorithm has typically become an *integrated sequence*—you and I could describe it, accurately, without needing to write it down (but we probably would need pencil and paper to *use* the algorithm to divide 1066 by 23; the paper isn't needed to remind us of the *procedure*, but to help us keep track of all the *numbers*).

Similarly, after you have driven to your brother's farm often enough, you could sit in your living room and *tell* someone how to drive there without needing to *see* the actual trees, or old red barns, or other landmarks. The *entire* sequence is now stored in your memory, and can be retrieved as a single "idea."

Of course, it is still *sequential*. You may be like the waitress who can recite the list of today's desserts—but when you ask her if they have pineapple pie, she has to go back, start at the beginning of the list, and watch carefully to see if she gets to "pineapple pie."

A still more reliable kind of knowledge goes beyond this sequential limitation. We turn now to this still-more-secure (or more mature) kind of knowledge, known as *frames*.

Frames

The idea of "frames" was introduced by Marvin Minsky [Minsky, 1975], and more-or-less simultaneously by several others. The word "frames," as used here, is a technical word, with a special meaning. This meaning can best be explained by considering two of the problems that "frames" were intended to solve, the "combinatorial explosion," and the mysterious source of additional information.

The *combinatorial explosion* is the name used to describe certain information processing tasks that quickly get out of hand, because the amount of information involved becomes so fantastically large. Suppose you want to translate a sentence from English into German. Perhaps the second word could have four different meanings—as "bow," say, could mean the act of bending forward from the waist, or could mean the front of a boat, or could mean a violin bow, or could refer to archery equipment (in fact, it has many additional meanings). Suppose the fifth word could have three different meanings. Suppose the eighth word could have five different meanings. Considering all possible combinations of these, there would be $4 \times 3 \times 5 = 60$ different sentences that could be constructed. Suppose, now that a similar situation held for the next sentence to be translated, with a possibility of 48 different meanings. For a paragraph of 6 sentences, there might be

$$60 \times 48 \times 60 \times 75 \times 60 \times 32$$

different meanings—but this number is equal to 2.48832×10^{10}, or 24, 883, 200,000—that is 24 billion, 883 million, 200 thousand *possible* translations. And, of course, if the first paragraph has over 24 billion possible translations, and the second paragraph *also* has 24 billion possible translations, the two paragraphs together have

$$24 \times 10^9 \times 24 \times 10^9 = 576 \times 10^{18},$$

or

576,000,000,000,000,000,000

different possible translations.

Clearly, things are getting out of hand. There is so much information here that we are being overwhelmed. It must be true that *we do NOT process information in this way, one small piece at a time*. There must be some larger "gestalts" that organize the possibilities, eliminating most of the really silly ones. This is the first problem that Minsky meant to solve by introducing the idea of *frames*. We turn now to the second problem.

The appearance of extra information was the other problem that *frames* are intended to solve. Suppose that some normally literate adults in the United States read the following paragraph:

> It was Paul's birthday. Jane and Alex went to get presents. "Oh, look," Jane said, "I'll get him a kite!" "He already has one," Alex responded. "He'll make you take it back."

After reading the paragraph, our typical readers will usually be able to answer questions such as these:

Q1: Why are Jane and Alex buying presents?
(Ans 1: Because it is Paul's birthday.)

Q2: Where did Jane and Alex go?
(Ans 2: To a store that sells, among other things, kites—a toy store, department store, or variety store.)

Q3: The next-to-the-last word in the selection was "it." What is the antecedent of this pronoun? [What did Alex mean, when he said "it" would have to be taken back?]

After answering these questions correctly, people are usually surprised to find that *not one of these answers is actually given in the paragraph itself.* [Before you tell me I'm wrong, please notice that nothing says that the first sentence provides the *reason* for the second sentence. Suppose the story had said: "It was raining. Jane and Alex went to get presents." Would you *then* say that they went to get presents *because* it was raining?]

In every case, we see information that seems to get *added* to the story, that somehow creeps in, seeming to come from nowhere.

The phenomenon is even more striking if you ask people to repeat the story a week or two later. Typically, they incorporate some of this additional information into the story *without being aware that they have added anything*. For example, they might say: "Jane and Alex went to the store to get presents." But the original paragraph never mentions a *store*.

Both the "combinatorial explosion," and the mysterious appearance of additional information, are explained by Minsky's *frames*.

Minsky hypothesizes that the information in your memory is organized into "bunches" or "clusters" called *frames*. When you read the sentence

It was Paul's birthday

you immediately recall ("retrieve from memory") the *birthday frame*. This frame contains a lot of information: it is the anniversary of the day Paul was born; maybe there's a party; maybe there's a cake; maybe the cake has candles on it; maybe there's one candle for each year; maybe Paul is supposed to try to blow out the candles; maybe there are invited guests; maybe the guests will bring presents; maybe the presents will be wrapped up in special fancy paper, with ribbons and bows; and so on.

The interpretation of the paragraph is now carried out *in relation to this "birthday frame."* That "extra" information, that isn't literally contained in the paragraph itself, *is contained in the birthday frame*. But we combine the information from the paragraph with information from the frame, not keeping them separate. That is why we can answer those questions—and, since this is what we *always* do with information that we hear or read, we are not aware of having done anything unusual. We don't think we "added" anything to the information in the paragraph.

But, of course, we did.

Notice that frames also protect us from *too much* information: in discussing the "birthday frame," I said it contained additional information, such as: "...maybe there are invited guests; maybe the guests will bring presents; maybe the presents will be wrapped up in special fancy paper, with ribbons and bows...."

Now, did the word "bow" mean:

(i) bend forward from the waist
(ii) what you use to play a violin
(iii) what you use to shoot an arrow
(iv) the front of a boat or else
(v) a decoration made up of loops of colorful ribbon.

Did you have to consider all six possibilities? No, because the birthday frame tells you to try, first, the "loops of ribbon" meaning, and go on to the others only if necessary. (Of course, you might have had clues to retrieve some *other* frames in addition to the birthday frame. Suppose the story said, "The other guests danced while Ann played square dance music on her fiddle"—or suppose it said "Paul's favorite present was a very large model sailboat"—and so on.)

Notice that knowledge which is organized as *frames* is quite different from knowledge that is organized as *sequences*. For one thing, frame knowledge is much more flexible. Unlike the waitress checking on pineapple pie, you don't have to start at the beginning and move forward one step at a time.

If you ask me which floor of my house my office is on, I don't have to start with the front-door and work ahead one room at a time. I can jump immediately to thinking about "my office." And in the birthday frame, you can jump around however you want to (or need to): you can start with the presents, or with "Happy Birthday" cards, or with what color the frosting on the cake probably is.

Frame knowledge is more secure than sequence knowledge, it is more flexible, and it is more complete. One of the important goals of effective math teaching is developing frame knowledge around the most essential concepts and techniques. Probably considerable experience, and sometimes the passage of time, is required for knowledge to become structured as *frames*. I have frame knowledge of my own home, and of my office, but I did not have it when I first moved in.

This book seeks to build a frame for the concept of *function*, a frame for *graphs*, a frame for *area*, and so on. As a result, students should feel "comfortable" and "at home" with these ideas—just as they do with the idea of birthdays.

Planning Knowledge and Heuristics

Someone who is "good at mathematics" is able to solve many problems that *nobody has told him how to solve*. Is this really surprising? It should not be. In nearly any other field we *expect* as much. We expect not merely that you can do what people have taught you how to do, but, by careful planning, we expect you can extend this substantially and go *beyond* the specific things you have been taught.

We can help students to go beyond what we teach them by showing them how to *plan*.

Suppose, for example, students know how to add

$$\frac{1}{2} + \frac{1}{2} = 1$$

$$\frac{1}{7} + \frac{2}{7} = \frac{3}{7}$$

$$\frac{2}{5} + \frac{2}{5} = \frac{4}{5}$$

and so on.

Suppose also that the students know about *equivalent fractions*, that $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots$, and so on.

Suppose, now, that the students face a *new* challenge. They need to add

$$\frac{1}{2} + \frac{1}{3} =$$

but nobody has told them how to do it.

Well, we need to make a plan. What can we do?

For one thing, we can ask *does this resemble any problem that we CAN solve*? Answer: yes, it does. It involves *adding fractions*, and problems such as

$$\frac{1}{2} + \frac{1}{2} = 1$$

or

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

or

$$\frac{1}{7} + \frac{3}{7} = \frac{4}{7}$$

would be problems we could solve very easily.

How is this new problem different? Answer: Well, in the easy problems, both fractions have the same denominator. In this new problem, the denominators of the fractions are different. O.K., then, *could we make this new problem more like the easy problems*? Answer: Well, we can try. We *do* know something about changing denominators. Maybe we could get the denominators to be the same. Let's see...

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \ldots$$

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \ldots$$

Aha! $\frac{2}{6}$ and $\frac{3}{6}$ have the same denominator. Now, $\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} = \frac{2}{6},$ so we can write

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$$

We have taken an *unfamiliar new problem*, worked on it, and turned it into a *familiar old problem* that we can easily solve.

What am I trying to say, here? Just this: if we carry students along with us, as we solve new kinds of problems, work out new methods, *plan*... then they will have a better chance of being able to do this sort of planning themselves, because they will have seen how we do *our* planning.

But if, instead, we just *tell* the students what the method is, then the students are likely to come to believe that they can only solve a problem if somebody has told them how to do it.

Whatever we need to do in mathematics, there is usually some *reason* why we need to do it. It helps students if we let them know what these reasons are, rather than proclaiming a method: "do it like this!"

Note: Questions or statements that guide us as we plan out how to attack some problem are often called *heuristics*. Thus, the question "*Does this (hard) NEW problem resemble any familiar old problem*?" is one *heuristic*. The question "How is this new problem different from the familiar old ones?" is another heuristic. It is worth letting students see how the skillful use of heuristics can make mathematics much easier, and can help to cause mathematics to "make sense."

In case they may be of use to you, here a few of the *heuristics* that I sometimes find helpful:

What *kind* of problem is this?
What problems does this *remind you of*?
If any, then:

How is this problem different?
How is it similar?
How can we make it *more* similar to these "familiar" or "easy" problems?

Can you break the problem up into several smaller problems? Can you solve any of these smaller problems? [Example: buying a car might be broken up into the problem of deciding you need to buy a car, the problem of deciding what kind of cars to consider, the problem of choosing, the problem of working out payments, the problem of getting it registered, the problem of getting it insured, etc. Maybe you can solve each of these *pieces* (or sub-problems) separately. Or, as a second example, dividing

$$21\overline{)1937}$$

might be seen as a problem of asking how often 2 goes into 19, trying out 9 x 21, finding out whether 9 x 21 is too large, subtracting 189 from 193, and so on. Maybe you *can* solve each sub-problem. But when you have solved *all* of the sub-problems, you have solved the original problem!]

What's *good* about this problem? How can we make use of this good feature? What makes this problem *hard*? How can we eliminate this obstacle? Or can we somehow work *around* this obstacle?

Can you make up an *easy* (or familiar) problem that is reasonably similar to this *new* problem? If so, solve the easy problem *and watch carefully how you do it*. Does that give you any clues as to how you might solve the new problem?

Find some *part* of the problem that you *can* deal with, *and do so*.

If you were asked to change this problem so as to make it easier, how would you change it? Does that give you any ideas?

VI. CRITERIA FOR DECIDING THE CHOICE OF TOPICS

Assume, for the moment, that you plan to use these math lessons in grade 3, 4, 5, or 6. You have decided to build a "3-ingredient" math program —skills, ideas, and applications—and you therefore want to include some of the key *ideas* of mathematics in your elementary school curriculum.

Which ideas do you select?* That is a very interesting question. We shall consider it presently. But perhaps there is a prior question: *What criteria should we use in selecting mathematical topics*?

Some important reasons *in favor* of selecting a particular mathematical topic for inclusion in your program are the following:

(1) Mathematics is a story that builds up gradually—it is often described as "cumulative." For example, one should not try to learn to add fractions until *after* one has learned something about what *addition* is, and something about what *fractions* are. The whole long story of mathematics is developed over many years of school work—perhaps from grades K through 12, often continuing on into college or even graduate school.

Consider some particular topic, which we can call Topic X. It is a strong reason in favor of including Topic X *if we need it in order to get on with the main development of mathematics*, that is to say, if

---

*The reasons for the selection would not be very different if one were considering grades 7 and 8 instead of 3 through 6.

Topic X is essential to the further continuation of the principle "story line" of mathematics.

For example, one surely has to learn to *count* before one can move ahead very far in the study of mathematics. (And, equally, a few years later one has to learn something about the concept of a mathematical *variable* if one is to follow the story of mathematics very far.)

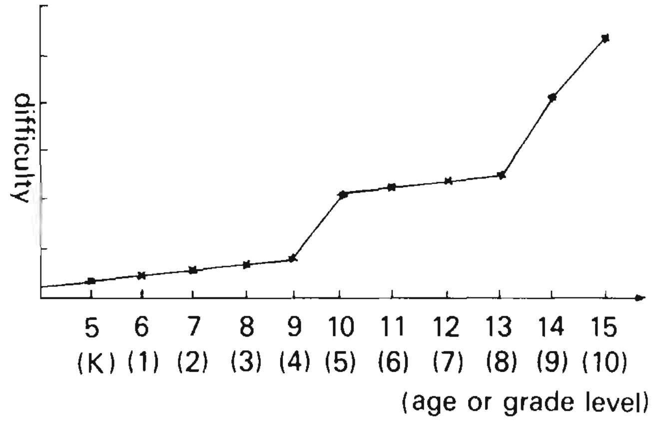
(2) A strong reason for including Topic X would be that *it is the kind of idea that takes a while to learn. Therefore, we must not delay too long in getting started.* Unfortunately, school programs often overlook this reason.

One or two examples will make our meaning clearer. The idea of *whole numbers* is very important, and probably takes a few years to learn. Fortunately, parents, baby sitters, schools, and even life itself all help young children learn about numbers and about counting. Children use small whole numbers to say how old they are, how many brothers or sisters they have, how many comic books they have collected, how many dolls they have, and so on. Chidren count on their fingers, they hide their eyes and count to 10 during games, they sing counting songs—there seems no end to the way children use counting and small whole numbers. As a result, the main ideas about whole numbers are usually learned well—children feel at home with "three" or "two" or "seven."

Contrast this with the case of fractions. Except perhaps for "one-half," children do *not* use fractions much. The idea remains strange to them. It should be no surprise that when, around grades 4 and 5, the school program attempts to deal with fractions as a major topic, *most children are not ready*. And since they are not comfortable with the basic ideas of what a fraction *is*, what it *means*. what it is *good for*, they are not ready to learn how to add fractions, or to multiply them, or to divide them. And unfortunately, there is abundant evidence that most children do not acquire much skill, nor much understanding, where fractions are concerned.

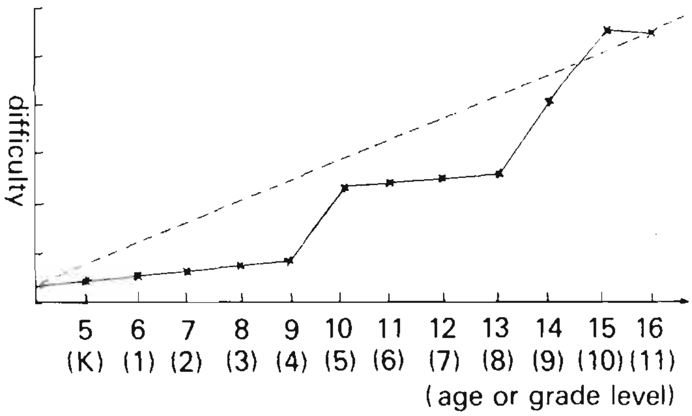
The hard ideas about fractions require careful development, over an extended period of time. But most schools typically fail to provide the years of experience that are needed.

The situation can be described by a graph:



From pre-kindergarten through (about) grade 3 or 4, one sees a careful, gradual development mainly of whole-number arithmetic. Hence, our "difficulty" graph slopes gradually upward. Unfortunately, fractions appear rather suddenly around grade 4 or 5, without adequate advance preparation. Again, at about grade 9, a sudden new difficulty appears – the use of variables. Again, this has not been prepared for adequately beforehand. Then, at grade 10, there is a sudden introduction of the ideas of proof. This, too, has not been preceded by a gradual buildup.

A more effective mathematics curriculum would *precede* each new step by *careful preparation beforehand*. The graphical picture of such a curriculum would look more like this:



The dotted line shows a curriculum that prepares carefully for each new topic. Consequently, the sudden jumps in difficulty that characterized the typical old curricula are eliminated: fractions, variables, and logical proofs have been gradually prepared for beforehand. (Cf. Davis, 1971-2.)

In short, a VERY STRONG reason for introducing Topic X EARLY, AND GRADUALLY, exists if Topic X requires a long, careful build-up. This implies that Topic X is of such a nature that it should not be neglected for years, then suddenly introduced abruptly.

(3) A third criterion should be: *does Topic X match the learning styles of students at this grade level?*

(4) A fourth criterion should be: is Topic X related to *interesting activities* that are appropriate for children of this age?

(5) Finally, a fifth criterion is: *can Topic X be introduced at this grade level in such a way as to suggest a true picture of the nature of mathematics?*

VII. UNDERSTANDINGS GROW GRADUALLY

The way we understand any particular thing will necessarily change over time. This is just as true of mathematical ideas as of any others. Unfortunately, this truth is often overlooked, and school programs sometimes set themselves the goal of giving a child "the correct idea" of things when the child first encounters them. This is not a sensible goal. The truth is that the child's understanding must develop gradually.

Consider the concept of *equality* and the symbol "=". For the pre-school or nursery child, "two things, and then two things more, makes four things," and, if one writes

$$2 + 2 = 4$$

the equality has a *direction* to it. (In fact, it might be better to write

$$2 + 2 \rightarrow 4$$

but nobody does write it this way.) The notation

$$2 + 2 =$$

will be seen as *posing a question*, and the *answer* will be 4. If you were to attempt to reverse this, and to write

$$4 =$$

most young children would be confused. To them it seems that you have given an answer, yet somehow you seem to be pretending that you have given a question.

Of course, a few years later the child meets more diverse experiences—perhaps hanging weights on a balance beam, for example—and then the child learns to deal with

$$2 + 2 = 4$$
or
$$4 = 2 + 2$$
or
$$3 + 1 = 2 + 2$$
or
$$3 + 1 = 1 + 3,$$

and so on.

Still later, when a student begins to deal with *mathematical logic*, the idea of "equality" changes once again. A statement

$$A = B$$

now comes to mean that A is the name of some mathematical entity, and B is the name of some mathematical entity, and, in fact, A *names the same thing that* B *names.*

Which idea should a student *begin* with? Clearly, the one he does—namely, 2 + 2 = 4 means that you put 2 things together with two things, and if you count the result, you have 4 things. The other ideas belong to later stages in a child's development, and ought to come along later, not at the very beginning. *At each stage of a child's life we ought to teach those meanings that are appropriate to that particular stage.* (One error of some "new mathematics" curricula a few years ago was to try to teach very mature versions to very young children, ignoring this process of gradual development.)

VIII. THE REASONABLENESS OF MATHEMATICS

Mathematics has been "invented," or "discovered," as *a reasonable response to reasonable challenges*. Counting developed so early in human history that the details are no longer known, but can anyone doubt that some need to keep track of the number of something-or-other was the challenge which inspired the invention? What could the "something-or-other" have been? Members of your family? Members of your group or clan? The number of tools? The number of times one has killed an animal on a hunt? The number of men needed to hunt a buffalo? It would be fun to know, but of course one can only guess. But surely counting was invented to meet some need, and it probably did the job fairly well. *Mathematics was surely a reasonable response to a reasonable challenge.*

The same has been true for every important advance throughout the history of mathematics. People needed to navigate boats safely, to survey the Nile delta, to build buildings, and so on. In each case some reasonable challenge led to the creation of appropriate mathematics. Mathematics continued to consist of reasonable responses to reasonable challenges.

I mention this because Madison Project teachers feel that it is important for students to view mathematics this way. If a student comes to regard mathematics as some sort of silly game that grown-ups waste time on, or some sort of artificial "schoolish" task, then we feel that we have failed. We want our students to recognize that mathematics consists of *reasonable responses to reasonable challenges*.

IX. THE CHOICE OF TOPICS

We return now to our earlier question: *which key ideas should be included in an intermediate-grade mathematics program?*

We would argue *in favor of including* ideas about

fractions
variables
equations
functions
negative numbers
graphs

and certain carefully-selected portions of geometry.

We would argue *against* including:

sets (except in certain cases)
the number-vs.-numeral distinction
certain topics in geometry.

Because I believe that the growing professionalization of teaching implies that teachers will come to have a larger role in making curriculum decisions — and must be ready to assume more responsibility in this area — I want to present briefly a few of the reasons that have convinced me to select these topics for inclusion. The reasons build on the criteria and assumptions discussed in the preceding sections.

The case for including VARIABLES.

Unfortunately, the name "variable" probably doesn't describe this important idea as clearly as one might wish. We can get a better notion of what a *variable* is from looking at some examples.

1) One well-known example is the familiar "x" of ninth-grade algebra. Now why do we use "x"? Usually because we need to name some number, and we are unable to use the ordinary kind of name such as "3" or "2001." Why are we unable to use a name like "3"? Usually for one of two reasons: either we *don't know* exactly what number we're dealing with, or else *we deliberately want to keep our options open.*

2) Another very familiar example might be the formulas for geometry:

Area of a rectangle: $A = b \times h$
Area of a circle: $A = \pi r^2$
Area of a triangle: $A = 1/2\,bh$
Perimeter of a circle: $P = 2\pi r$

and so on.

This example illustrates why one wants to keep one's options open. If we have a circle of radius 6 inches, then we know that

$$\pi \cdot 6^2 = \pi \cdot 36 = 113 \text{ square inches.}$$

But we don't want to be restricted to circles with a radius of 6 inches. We want to be able to compute the area of *any* circle. The formula

$$A = \pi r^2$$

allows us to do this!

3) If we want to write an expression to stand for all the *even* integers, we can write $2 \times n$, where *n* is any integer; odd integers are just

$$(2 \times n) + 1$$

where n is any integer.

Now, should the concept of *variable* be included in the curriculum for grades 4, 5, and 6? *We would argue: yes, it should be included.* Why?

(i) For one thing, one *cannot* progress with graphs, functions, equations, etc., *without* the concept of variables. So it immediately passes one test: *we need it in order to get on with the main themes in the unfolding story of mathematics.*

(ii) For another thing, the concept of variable represents one of the biggest obstacles in most school curricula; it accounts for the "cliff" at (about) grade 9. Variables are traditionally ignored for about eight years of school, then, at grade 9, *the entire year's work is based on the use of variables.* But no readiness has been created, no advance preparations have been made. The idea has not been allowed to grow gradually in the student's mind. Rather, it is suddenly and traumatically thrust upon unprepared students — and, as a result, most students find 9th grade algebra unnecessarily difficult.

All of this can be avoided by developing the idea of variable carefully and naturally, over a period of several years prior to grades 8 or 9.

(iii) Are there appropriate activities for children this age that present the idea of *variables*? Yes, there are a great many. It is no exaggeration to say that the majority of children have *enjoyed* the activity presented in Chapter 3. [You can observe children making use of this activity in the film *First Lesson**. The film leaves no doubt that the children are enjoying it.]

SUMMARY: The case for including *variable* in the intermediate-grade curriculum seems to us to be extremely strong. It is one of the most important ideas to include at this grade level.

NOTE: In introductory work, we use the notation "□" and "Δ" instead of "x" and "y" to represent variables. This seems to work much better with beginning students at almost any grade level. (One interesting discussion of this is presented in Cetorelli, 1979. See also Chapter 1.)

Which Parts of Geometry Should Be Included? There has been substantial disagreement about which geometric topics and methods to include in grades 4, 5, and 6. To us, the answer seems reasonably clear.

What "parts of geometry" are there, anyhow? Before we can select the topics and methods to include, we need to ask: what methods and topics are there? The list includes at least the following:

Euclidean synthetic geometry. This is the traditional geometry of grade 10. It is based on careful verbal definitions, careful statements of theorems, and careful proofs based on the inference schemes of mathematical logic. The subject is very precise, highly verbal, and often quite complicated. Furthermore, the demands of the logic are often quite different from the "feeling" of the geometric figures themselves.

We consider this an *unsuitable* topic for study at the earlier grades, primarily because its precise verbal nature does not match the cognitive preferences of younger children—they tend to find the precision gratuitous, a mere matter of being foolishly finicky.

"Cartesian" or "analytic" geometry. In one of its simpler manifestations, this includes the topic of *graphs*. Graphs are of great value in nearly all applications of mathematics. One excellent graph appears on the inside back page of every issue of the *Wall Street Journal.* This graph shows a great deal about the stock market, in a form that can be taken in at a glance.

The Madison Project, The British Nuffield Mathematics Project, and several other groups have developed and tested a great many activities related to graphs that are highly suitable for use by children of this age. We would argue that the importance of this topic, and its clear suitability for this age child, are strong—indeed, decisive—arguments in favor of including *graphs* in the curriculum for grades 3-6.

Vector geometry. This could well be another strong candidate. For modern applications, vector geometry is of the greatest importance. Moreover, since its fundamental concepts deal with, essentially, "taking one giant step forward" and "taking one giant step to the right," it would seem to match a child's typical perception of space and motion. Surprisingly, however, nobody seems to have developed any lessons in *vector geometry* that are appropriate to children in the intermediate grades.

Computational geometry. This approach to geometry is based upon "moving one step forward," "turning to the right," and counting. These are all very natural activities for children. It is no surprise that excellent lessons in computational geometry have been developed for grades 3 through 6, primarily as a result of the work of Seymour Papert at M.I.T. Unfortunately, most of the best lessons of this sort require access to computers, and are not yet readily available to most schools and homes.

Geoboard geometry. A "geoboard" is a square board, often of plywood, with regularly spaced nails driven part way in. Geometric shapes are made by stretching rubber bands over the protruding heads of the nails. There are many interesting parts of geometry that can be presented to children of this age by means of enjoyable and effective activities. This is an ideal part of geometry for inclusion in the intermediate grade curriculum.

Special topics in geometry. This includes Marion Walter's "milk-carton cutting"(which involves 3-dimensional visualization), the E.S.S. "mirror cards" (also by Marion Walter), uses of the oriental tangram constructions, polyomino problems (and other problems in tessellations).

Topology. Topology has been described as the kind of geometry you could study if your diagrams had to be drawn on the side of a rubber balloon. You could *not* study size, length, or area, because as the balloon expands (or loses air and shrinks) the sizes, lengths, and areas keep changing. You

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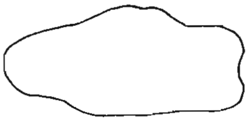
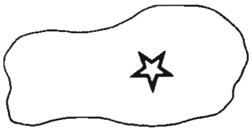
*For information on Madison Project films, write to the author at 1210 West Springfield Avenue, Urbana, Illinois 61801.

could not speak of "straight" lines, because as the rubber expands or contracts, distortions creep in. You *could*, however, distinguish a *closed curve*



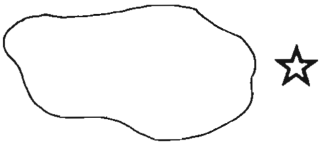
from a curve which was *not* closed (we assume the balloon doesn't actually break!).

Also, you could distinguish the *inside* of a simple closed curve:



The star is *inside* the curve.

from the *outside* of the curve:



The star is *outside* the curve.

Probably because Piaget found young children were strongly aware of such topological properties as *inside* and *outside* there has been some tendency to introduce topological ideas into elementary schools. We would argue *against* this. Topological ideas fail most of the tests we have proposed; for example, probably no students find their onward progress blocked by their ignorance of topological knowledge. What they need to know, nearly all students learn spontaneously—there is no need to teach it.

Sets. This book uses sets, *but only where they seem to be helpful*—specifically, we consider the set (or "collection") of whole numbers that would make this inequality *true*:

$$3 < (2 \times \square) - 1 < 8$$

—and other problems of this type. The set just mentioned is in fact $\{3, 4\}$ which is to say that if you write "3" in the "□", you get a *true statement*:

$$3 < (2 \times 3) - 1 < 8$$

i.e., $3 < 5 < 8$), or if you write "4" in the "□" you get a *true statement*:

$$3 < (2 \times 4) - 1 < 8$$

i.e., $3 < 7 < 8$), but if you write any *other* whole number in the "□", the result will be a *false* statement, as in this case:

$$5 \rightarrow \square$$
$$3 < (2 \times 5) - 1 < 8 \qquad \textit{False.}$$

We do *not*, however, use sets with younger children—in grades K, 1, or 2, say—as an introduction to whole numbers. Such a use of sets fails most of our criteria for inclusion in the curriculum. In particular, *counting* is a very natural activity for young children. Sets, by contrast, involve complications that are not at all helpful to 4 or 5 year olds.

The human race used counting, reliably and extensively, for at least 4,000 years before Georg Cantor, in 1874, invented "set theory." Cantor's reasons for introducing *sets* were, as usual, quite reasonable, *but they had no relation at all to the job of helping young children to learn about numbers.*

SUMMARY. Our conclusion (after applying our various criteria) is that sets are useful, in a small way, in relation to certain mathematical problems in grades 3 through 12. Sets are probably NOT useful as a way to teach young children their first ideas about numbers, and we do not use them for this purpose.

The task of choosing topics for inclusion in the curriculum is a most important one. The criteria for choosing, presented in the preceding sections, have been the basis for the Madison Project choices. I hope these criteria can help you in your own decision making.

X. THE USES OF MATHEMATICS

We have said earlier that a strong mathematics program needs to contain three ingredients: (i) the usual *skills* of arithmetic; (ii) some of the important basic *ideas* of mathematics (such as "function," "variable," "graph," etc.); and (iii) some interesting *uses* of mathematics. The present book assumes that the usual arithmetic skills are well provided for in your present mathematics program, and will continue to be. We do not present material for "Ingredient Number 1" in this book, although the *practice* and *motivation* that can be obtained in the other two strands can result in improved performance in the area of basic skills.

This book, in Chapters 1 through 50, presents an introduction to some of the most important *ideas* of mathematics. In that sense, this book deals almost exclusively with the second of the three ingredients, leaving it up to teachers or parents to provide the other two ingredients from other sources.

Applications, however, are so important that we present here a few possibilities that have proved especially effective.

Descriptions of reality. The main point of the third strand is to give students abundant experience with the relation between *reality* and *mathematical descriptions of reality*.

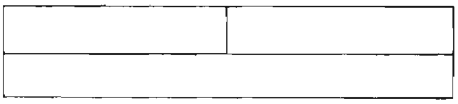
1. *Cuisenaire rods for fractions.* About 20 years ago, I was introduced to Cuisenaire rods, for which I am eternally grateful. The first use of rods that I found myself making involved fractions. There is abundant evidence that most students do not ever learn to deal confidently, easily, and correctly with fractions. This is unfortunate, because most of high school mathematics depends upon fractions; so do more advanced subjects such as calculus and statistics. A student who is weak in his dealing with fractions will find that he has a persistent handicap.

First Method. I have two ways of using Cuisenaire rods to help develop ideas about fractions. In the first method, I show the students a light green rod on top of a dark green rod, with the left ends flush:

I tell the students: "The light green rod is half as long as the dark green rod. Can you find *some other* pair of rods where one rod is half as long as the other?" The answers, which children will nearly always get, are:

The white rod is half as long as the red rod.
The red rod is half as long as the purple rod.
The purple rod is half as long as the brown rod.
The yellow rod is half as long as the orange rod.

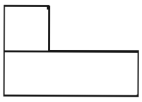
Proof. At some point in this discussion, I pose this challenge: "Suppose I didn't believe you. What could you do to *convince* me that (say) the yellow rod is half as long as the orange rod?" The answer, nearly always forthcoming, is to put *two* yellow rods together on top of one orange rod:



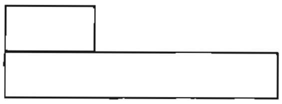
Of course, thus far we haven't done very much, since nearly all children feel at home with "one half." But now we have established a format, and we can move into new territory:

"Can you show me a rod that is *one third* as long as another rod?"

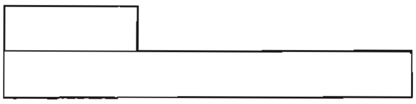
Answers:



white over light green



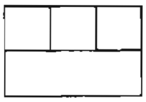
red over dark green



light green over blue.

"If I doubted that, how could you convince me?"

Answer: Add the other two rods; e.g.,

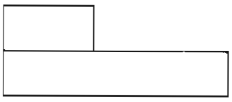


3 whites on a light green rod.

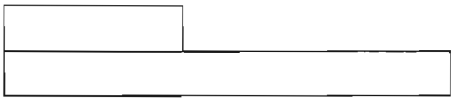
You can proceed as far as you like. For example:

"Show me a rod that is *two fifths* as long as some other rod."

Answers:



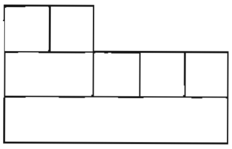
red rod on a yellow rod



purple rod on an orange rod.

"How could you prove that?"

Answer: Usually children will use white rods (for the "red on yellow" case) or else red rods (for the "purple on orange" case. E.g.:



The white rods show the 2-to-5 ratio

Second Method: I have a second way of using Cuisenaire rods to give children experience with fractions. I choose some rod—red or light green are good choices—and say: "If I call the *red* rod "one," which rod should I call "two"?"

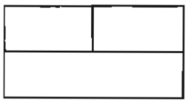
NOTE: There are two correct answers to this question, because there are two different mathematical structures that can be matched up with reality. The "counting numbers" 1, 2, 3, 4,...have two separate properties that can be used: their *size*, and their *sequential order*. These properties are independent—one could have *order without size*, as in the case of letters of the alphabet, or one could have *size without order*, as in the case of vectors in three dimensions.

If students match the rods against the *order* structure of the numbers, then if red is one, the *next* number is *two*, and the *next* rod is *light green*. *This answer is not wrong, but it is NOT VERY USEFUL*. If a student answers "light green," I say "Try to do it another way," and that usually suffices to elicit the other answer.

The answer I really want is: purple. "If red is called one, the purple rod should be called two." This matching of reality to mathematics makes use of the *lengths* of the rods and the *size* of their numbers; consequently, it preserves *addition*. For example,

$$1 + 1 = 2$$

now corresponds to
red + red = purple.



After we once establish that this is the match that we want, all the rest of the work usually proceeds easily.

The names of the various rods are then:
the dark green rod is called "3"
the light green rod is called "1 $\frac{1}{2}$ "
the brown rod is called "4"
the white rod is called " $\frac{1}{2}$ "
the yellow rod is called "2 $\frac{1}{2}$ "
the orange rod is called "5"
the black rod is called "3 $\frac{1}{2}$ "
the blue rod is called "4 $\frac{1}{2}$ "

I have purposely NOT listed these in order; in working with students I have learned to be careful to avoid obvious orders. If we went from red to light green to purple to yellow...and so on...then some children may merely go by the sequential order, 1, 1 $\frac{1}{2}$, 2, 2 $\frac{1}{2}$ ...and may consequently fail to see the relation between the reality and the mathematics. That is to say, after 1, 1 $\frac{1}{2}$, 2, 2 $\frac{1}{2}$, you could answer 3 *just by thinking about the numbers, without thinking about the rods at all.* But our main point is to have the students *carefully thinking about the match between the reality and the mathematics.*

After we have completed the "if red is one" game, we can say:

"If light green is one or 'if the light green rod is called one', then which rod is called 'two'?"

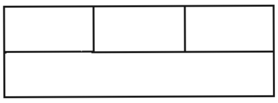
"Can you tell me the names of any of the other rods?" Asking the question this way avoids the problem of our establishing a misleading sequential order.

Addition of fractions. Suppose we want to add

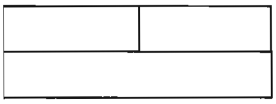
$$\frac{1}{2} + \frac{1}{3}$$

We must first decide which rod to call *one.* (I usually leave this as a problem for the children; in order to have a rod called $\frac{1}{2}$, they must give the name "one" either to the *red* rod, or to the *purple* rod, or to the *dark green* rod, or to the *brown* rod, or else to the *orange* rod. But, in order to have a rod named " $\frac{1}{3}$ ", it is necessary to give the name "1" either to the *light green,* or to the *dark green* rod, or else to the *blue* rod. Since we must have *both* a rod named $\frac{1}{2}$, *and also* a rod named $\frac{1}{3}$, there is an inescapable conclusion: we must give the name "1" to the *dark green* rod. Then the *red* rod is named " $\frac{1}{3}$ ", and the *light green* rod is named " $\frac{1}{2}$ ":

If dark green is one, then red is one-third.



If dark green is one, then light green is one-half.



We can now carry out the addition

$$\frac{1}{2} + \frac{1}{3}$$

as follows: we first put a light green rod ($\frac{1}{2}$) and a red rod ($\frac{1}{3}$) on top of a dark green rod:

It now remains only to figure out what name to give to our answer; this is easily solved by using white rods. The result (of course) is that the answer is $\frac{5}{6}$.

Division: One of my favorite problems is to use Cuisenaire rods to explain the *division of fractions.* Most adults do not understand the meaning of, say,

$$\frac{1}{3} \div \frac{1}{2}$$

Adults may have memorized a rule to "invert and multiply," so they may be able to write

$$\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$$

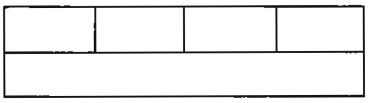
but they usually *cannot* exlain *what this means.*

We can approach this problem with the valuable heuristic of *thinking of a similar, but easier, problem.* Let's select

$$8 \div 2$$

This is *similar,* in the sense that it's still of the form A ÷ B, but we have *eliminated the fractions* (which makes the problem easier), and we have selected the numbers 8 and 2 *so that the problem "comes out even"—the answer, also, is a whole number.*

The question 8 ÷ 2 translates into the Cuisenaire rod problem: "How many red rods fit on top of a brown rod?" This, of course, is easily answered:



4 red rods fit on top of a brown rod, so the answer is

$$8 \div 2 = 4$$

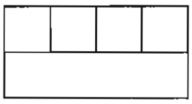
We can move gradually toward $\frac{1}{3} \div \frac{1}{2}$, by considering next a problem such as

$$2 \div \frac{1}{2}$$

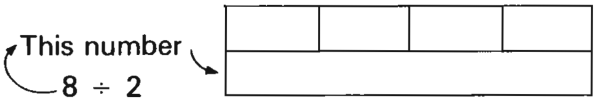
We can call the red rod "1," so this division problem translates into "How many white rods fit on top of a purple rod?"

The answer, of course, is 4; hence we have

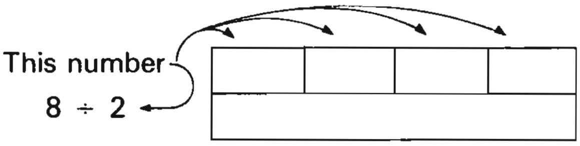
$$2 \div \frac{1}{2} = 4$$



Perhaps we are now ready to tackle $\frac{1}{3} \div \frac{1}{2}$. We need to call the *dark green* rod "1,"; then light green is $\frac{1}{2}$, and red is $\frac{1}{3}$. In order to translate into "rod language," we can study VERY CAREFULLY how we solved the 8 ÷ 2 problem:



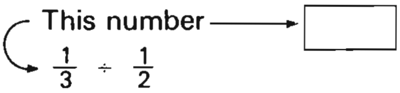
is represented by the rod ON THE BOTTOM.



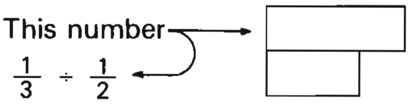
is represented by the rod (or rods) ON THE TOP.

All right—now let's preserve this pattern carefully, and apply it to

$$\frac{1}{3} \div \frac{1}{2}$$



is represented by the rod ON THE BOTTOM (which, therefore, should be a *red* rod).



is represented by the rod (s) ON THE TOP (which, therefore, should be a *light green rod*).

The question now is: *"How much of the light green rod fits on top of the red rod?"*

and—BEHOLD!—The answer is: $\frac{2}{3}$. So the "invert and multiply" rule DOES produce *meaningful* answers!

Once we have seen how this works, we can easily make up other examples.

Instead of interpreting 8 ÷ 2 as red rods on top of a brown rod, we could imagine a board 8 feet long. How many 2 ft. pieces could we cut from this 8 ft. board? Four, of course.

If we try to use *this* model for $\frac{1}{3} \div \frac{1}{2}$, it won't work, for a very important reason: if we want "2 ft. pieces," *then anything less than 2 feet probably won't do.* Had the long board been $8\frac{1}{2}$ feet long, we would still have gotten only 4 "2-foot pieces." Situations like this are called *discrete* (or, sometimes, "quantized"). Only certain widely-spaced numbers are acceptable (often, only whole numbers), and numbers "in between" these are NOT acceptable. If $300 buys an airplane ticket for one person, from New York to London, how many people can travel for $900? Answer: three. But how many can travel for $875? Answer: two. Of course, there would be $275 left over, but that is not enough money to purchase another $300 ticket.

The opposite of "discrete" is *continuous.* The number of passengers in an airplane is discrete; so is the number of books you have, or the number of automobiles. But the amount you weigh is *continuous*; so is the distance you can travel on the amount of gasoline in your car, or the length of time you have to wait to get a table in a restaurant.

Since the answer to $\frac{1}{3} \div \frac{1}{2}$ is not a whole number, we cannot use *discrete* examples; we must think of a *continuous* example. Here is one possibility: if it takes half a tankful of gas to drive to Chicago, how far can you get on $\frac{1}{3}$ of a tank? Answer: you could get $\frac{2}{3}$ of the way there.

2. *Cuisenaire rods for area and volume*. Cuisenaire rods give us an effective way to help children learn about *area* and *volume*. Here, too, we start with the discrete case (which is the only one we shall consider in this section).

For area, we want a clear, concrete question, *without using the word "area."* Here is my favorite: show the students one yellow rod, and say: "Suppose I have a stamp pad with blue ink, and I want to use a white rod as my stamp. I want to make this yellow rod blue, by stamping every bit of it. What is the smallest number of times I can stamp, in order to do it?"

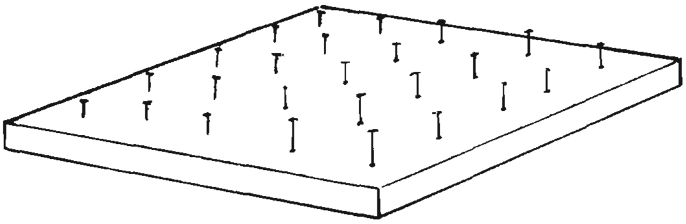
Answer: all together, 22 times (five stamps on each long face, plus one stamp on each end). *Now* we can tell the children that this is called the *area* (or "total surface area") of the yellow rod.

Volume: To get a similarly concrete question for volume, we can say that we want to glue white rods together to make a yellow rod (of course, after gluing, we have to paint it yellow). How many white rods should we glue together?

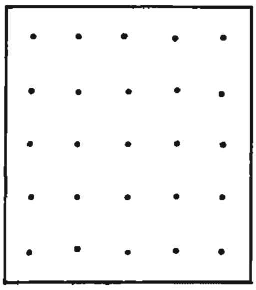
Answer: 5. (Again, after using these "concrete" questions, we can tell the students that this "5" is the *volume* of the yellow rod.)

Metric measure. In both of the preceding examples, the students have, in fact, been learning *metric measure* as well as "area" and "volume," because the area of one face of the white rod is *one square centimeter*, and the volume of the white rod is *one cubic centimeter* ("c.c." or "cm.^3").

3. *Geoboard geometry*. A "geoboard" is a square piece of wood, with nails pounded partway in. It looks like this:



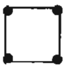
Or, looking straight down at it, it looks like this:

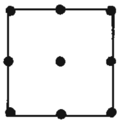


My favorite size has five rows and five columns of nails, with a two inch separation between nails. It is very helpful if the outside rows and columns of nails are exactly 1 inch from the edge of the board (this means the board is 10 inches by 10 inches). In that case, several boards can be placed together to make a larger geoboard, and the spacing of the nails remains regular. The nails must be located quite precisely, or false relationships will appear. One way to get geoboards is to make a paper pattern, and ditto it up, giving each child a pattern, which can be taped onto the wood, and nails can be driven through the dots on the paper. Even if you do not use such paper patterns to make geoboards, you still want to ditto up a large supply of papers with dots corresponding to nails. We call this "dot paper." Patterns (on figures) are made by stretching rubber bands over some of the nails; important figures can be preserved by copying them onto the dot paper.

It is also useful to have a supply of 2-inch squares, cut from slightly stiff colored paper. Here are some typical geoboard activities:

(i) "Can you make a square?" (Many possible answers, including:





and so on.)

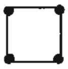
(ii) "What is the *smallest square you can make?" "What is the largest* square you can make?"

(iii) "Can you copy these squares onto dot paper?"

(iv) "Can you make a triangle?"

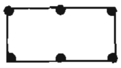
(v) "Can you make a rectangle?"

(vi) "I am going to call this *area one*"



"Can you make a rectangle with *area two*?"

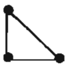
Answer:



The colored paper squares can also be used as units of area, to help the children see how "area" is determined.

(vii) "Can you make a rectangle with area three?" "Four?" etc.

(viii) "What is the area of this triangle?" Showing



Answer: one half; doubt can sometimes be overcome by folding one of the paper squares along a diagonal, and cutting it into two congruent triangles.

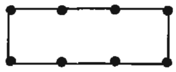
(ix) "Can you make a triangle with area *one*?" One and a half?" "Two?" etc.

(x) "Make whatever shape you want, and see if the other students can find what the *area* is."

There are three main methods that can be helpful:

First Method: See if the shape is *half* of something you already know: Thus, the area

must be $1\frac{1}{2}$, because it is half of



and half of 3 is $1\frac{1}{2}$.

Second Method: Figure out the area *outside* of the shape. Thus, to find the area

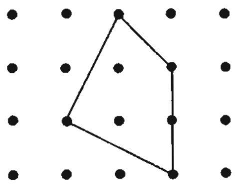
we see that we could start with an area of $1\frac{1}{2}$

and remove from it an area of $\frac{1}{2}$

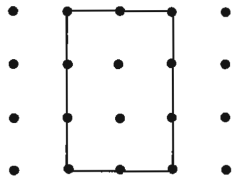
so that the area of

must be $1\frac{1}{2} - \frac{1}{2} = 1$

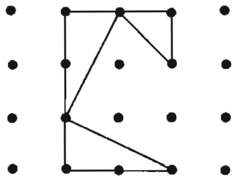
Second example: This area



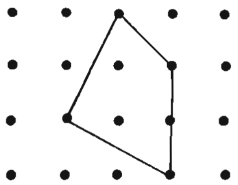
can be found by starting with



which has area 6, and removing three pieces

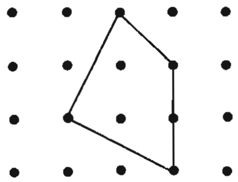


The areas removed are: 1, $\frac{1}{2}$, 1, for a total of $2\frac{1}{2}$
Hence, the area of

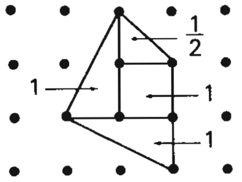


must be $6 - 2\frac{1}{2} = 3\frac{1}{2}$.

Third Method: Figure out the area inside the shape by dividing it up into pieces that you can recognize. We can solve the preceding problem by this method:



Divide it up:



So the total area is $1 + 1 + \frac{1}{2} + 1 = 3\frac{1}{2}$

With a little practice, you and your students can become quite proficient in finding the area of various (possibly wierd) shapes.

(xi) Here is a lovely task, which I learned from Donald Cohen: write on the board

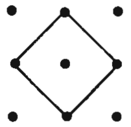
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.

Say: "The smallest square you have made had area 1. The largest had area 16 [on a 5-nail-by-5-nail board]. Can you make a square with area 2, a square with area 3, and so on? As you find them, we'll check off the numbers."

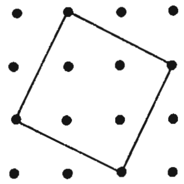
Students easily find squares with areas 1, 4, 9, and 16. They may decide there are no others. This is useful, and you can point out that the numbers 1, 4, 9, and 16 are *called* "squares." (You can also practice with numbers like 49, 81, 100, 121, 144, 169, and 196.)

But, in fact, you *can* make some more squares. Be careful that the children don't make *rectangles* that are *not* squares . To qualify, each shape *must be a square.*

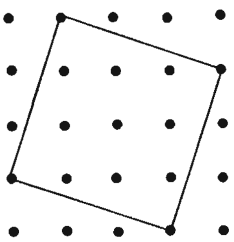
Here are the other possibilities:



Area 2 (either $4 \times \frac{1}{2}$, or else $4 - (4 \times \frac{1}{2})$



Area 5 either $(4 \times 1) + 1$, dividing up the *inside*, or else $9 - (4 \times 1)$, figuring out the *outside*.



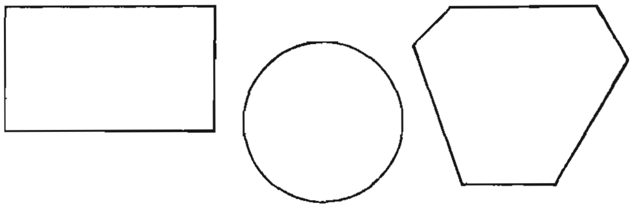
Area 10 inside: $4 + (4 \times 1\frac{1}{2}) = 10$; or from the outside, $16 - (4 \times 1\frac{1}{2}) = 16 - 6 = 10$.

These 3 shapes are skewed off a bit, but each is a perfect square. If you have studied analytic geometry, you can easily prove this by using the law of negative reciprocal shapes to establish that each angle is a true right angle.

(xii) A plane figure is called *convex* if *any two points of the figure* can be joined by a (straight) line segment that consists only of points of the figure. Roughly speaking, a figure is *convex* if it has no holes in it, and no "bites" that have been chewed out of it.

Not convex:

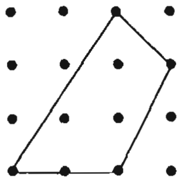
Convex:



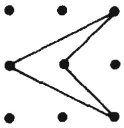
Now, here is a lovely challenge using a 5-nail-by-5-nail geoboard:

"Can you make a 3-sided convex figure?" (Easily; any triangle qualifies).

"Can you make a 4-sided convex figure?" (Again, easy: any square qualifies; any rectangle does; any parallelogram does; any trapezoid does; even this figure does:



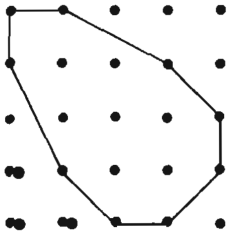
In fact, this was *so* easy that you might suspect that perhaps *every* four-sided figure on a geoboard is *convex*, but it isn't so as illustrated by this shape:



"Can you make a 5-sided convex figure?" Answer: yes.

Now, here's the really interesting question: "On a 5-nail-by-5-nail geoboard, what is the *largest* number of sides that a *convex* figure can have?"

The answer is very surprising: it is 9, because of the figure



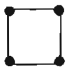
(xiii) There is a useful *guessing game* with geoboard shapes. One pupil—Leslie, say—makes a geoboard shape, *and does not let anyone else see it*. The task is for the class to ask questions, which Leslie answers, until someone is able to duplicate Leslie's figure. (You can adjust the precise rules to suit your class. You may want an *observer* to see Leslie's figure, to check that Leslie is answering questions correctly. You may want to restrict the kinds of questions that can be asked, perhaps to just "yes or no" questions, or perhaps to allow questions only where the answer will be "yes," "no," or some number.)

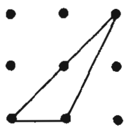
Here are some useful questions:
How many sides does it have?
Is it convex?
Is it a triangle?
Is it a trapezoid?
How many nails does the rubber band touch?
How many nails are inside the rubber band?
How many of these do NOT touch the rubber band?
What is the *area*?
Does the rubber band touch any nails in the top row across?

(xiv) *Pick's Theorem*. I like to pose this challenge to students:

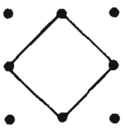
"Suppose you have heard a rumor that someone has found a formula that will tell you the *area* of a geoboard figure, if you know the *number of nails inside*. Now, like most rumors, this is a bit vague in details. How could you check up to see if the rumor might be correct?"

There are many possible ways to proceed. First, let's see if we can find *different areas* with the *same* number of nails inside. Consider these:





Each has 4 nails inside, and each figure has *area 1. Maybe the rumor is true*! Consider these figures:

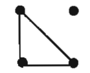


Both figures have 5 nails inside, but one figure has area 2, whereas the other figure has area 1 $\frac{1}{2}$. *So it seems that the rumor must be false*! After all, if I said to you "The figure has 5 nails inside," you wouldn't know whether to tell me "Therefore the area must be 2" or whether to say "Therefore the area must be 1 $\frac{1}{2}$."

Aha! But wait a moment! The rumor is a bit vague about "inside." Maybe we need to distinguish "boundary nails", which the rubber band touches, from "interior nails," which are inside but do NOT touch the rubber band. Let B stand for the number of boundary nails, and let I stand for the number of interior nails. Let A be the area. The figures we have made thus far show this pattern:

B	I	A
4	0	1
4	1	2
5	0	$1\frac{1}{2}$

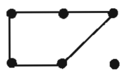
Perhaps we should look at more shapes, very systematically. To get started in an easy way, let's avoid interior nails. The shapes gives us the B, I, and A values shown.



I = 0

B = 3

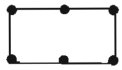
A = $\frac{1}{2}$



I = 0

B = 5

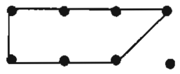
A = $1\frac{1}{2}$



I = 0

B = 6

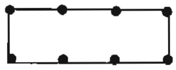
A = 2



I = 0

B = 7

A = $2\frac{1}{2}$



I = 0

B = 8

A = $3\frac{1}{2}$

It looks as if, with I = 0, every time we increase B by 1, *the area increases by* $\frac{1}{2}$.

Maybe the formula is something like this

$$A = a + bB + cI,$$

and we need to find the constants "a," and "b," and "c." It seems that b must be $\frac{1}{2}$, so we have

$$A = a + \frac{1}{2}B + cI.$$

Can we find the number a? Well, if I = 0, and B = 3, then A = $\frac{1}{2}$, so we have

$$\frac{1}{2} = a + (\frac{1}{2} \times 3)$$
$$\frac{1}{2} = a + 1\frac{1}{2}$$

and *a* must be negative one. So the formula is

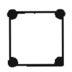
$$A = \frac{1}{2}B + cI - 1.$$

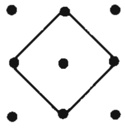
Now we need to find *c*. If we compare

I = 0

B = 4

A = 1





I = 1

B = 4

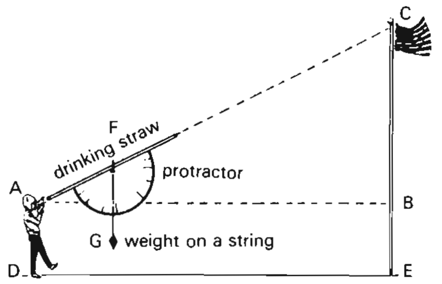
A = 2

it seems that *increasing I by 1* had the effect of *increasing A by 1*. So probably the formula should be

$$A = \frac{1}{2}B + I - 1$$

We haven't really *proved* it completely, but if you try this formula—known as *Pick's Theorem*—on many different shapes, I think you will find it very useful. (Cf. Ewbank, 1973; Hirsch, 1974; Laing, 1979.)

4. *Height of the school flagpole.* The task is for the students, usually working in small groups of two or three students per group, to invent and use some method to determine the height of the flagpole in the schoolyard (or the height of some ornament on the school building, or something of the sort). The method most commonly used by students is to make a careful scale drawing. This requires measuring an angle. One way to do this is to take a drinking straw and tape it to a protractor, then to look through the straw at the top of the flagpole:

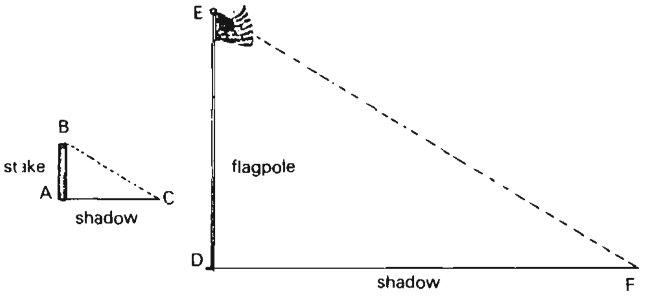


The angle A F G has the same measure as angle A C B, and its measure can be easily read off from the protractor by using a weight G hanging vertically on a string. The distance DE—from the student to the base of the flagpole—must be measured directly, and so must the distance AD, the height of the student's eyes above the ground. A careful scale drawing can now be made, perhaps with a scale of one inch on the drawing corresponding to 1 yard (or 1 meter) in the reality. The distance CB (or CE) can now be measured *on the drawing*, and converted, by using the one-inch-means-one-yard scale, to the correct distance for the actual flagpole.

Other methods are sometimes proposed, especially these:

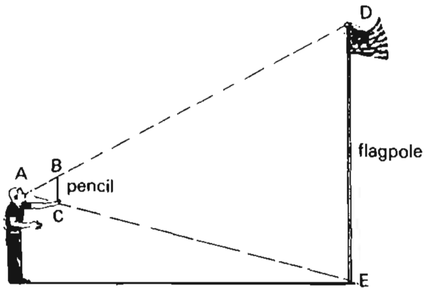
Second Method: Put a post in the ground, say three feet tall above ground. Wait until that time of day when the shadow of the post is exactly as long as the post is. At that moment, mark the tip of the shadow of the flagpole. Measure the length of the shadow; that is the height of the flagpole.

Third Method: Drive a stake in the ground, as in the Second Method. The triangle made by this stake and its shadow *is itself a "scale model" of the flagpole and its shadow*. Hence use it as in the case of the scale drawing in Method 1.



(Triangles ABC and DEF are *similar*. Hence, $\frac{AB}{AC} = \frac{DE}{DF}$. But AB, AC, and DF can be measured directly; the equation can then be solved to give DE.)

Fourth Method: Students with training in *art* may use a modified version of this method, holding a pencil at arms length:



Again, triangle ABC and triangle ADE are similar triangles.

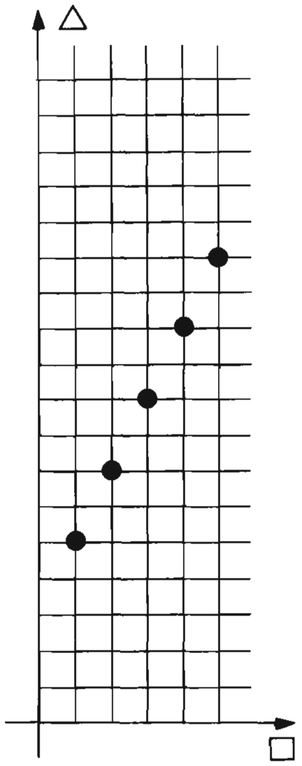
5. *Functions*. A function may be represented by a *table*

□	△
1	3
2	5
3	7
4	9
5	11
•	•
•	•
•	•

or by a *formula*

$$(\square \times 2) + 1 = \triangle$$

or by a *graph*

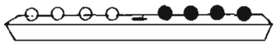


The ideas of function, graph, etc. are developed later in this book. For the moment, we consider only where *functions* may come from:

(i) Linear and non-linear elasticity: examples are given in Chapter 49.

(ii) The *Shuttle puzzle*. This puzzle is available from World Wide Games, Delaware, Ohio 43015 (or you can make it yourself). For the full puzzle, the starting position is

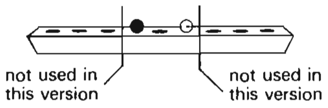
and the goal position is



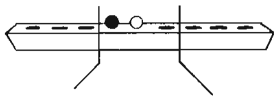
There are two kinds of legal moves: (i) a marble may move to an unoccupied adjacent space; or (ii) a marble may jump over one marble of the opposite color. The black marbles can move *only* toward the right, and the white marbles can move *only* toward the left.

The complete puzzle, by itself, does not generate a function. To get a function, we consider various modified versions of the puzzle:

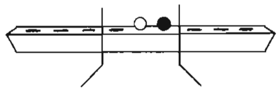
First, we use only one black marble, one white marble and only the middle three slots in the puzzle:



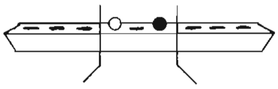
starting position



white marble moves left



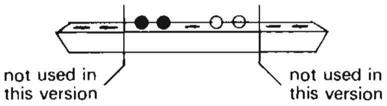
black marble jumps over white marble



white marble moves left

The solution required 3 moves.

The next version of the puzzle uses 2 white marbles, 2 black marbles, and the middle 5 slots of the puzzle. The starting position is:



Solution: white moves left; black jumps; black moves right (this is the key step; if you omit it, you will not be able to solve the puzzle); white jumps; white jumps (again); black moves right; black jumps; white moves left. The solution requires 8 moves.

In the next version, we use 3 black marbles, 3 white marbles, and the middle 7 slots in the puzzle.

Solution: white moves left; black jumps; black moves right (one of those key steps!); black jumps; black jumps again; black jumps for a third time; white moves left; white jumps; white jumps again; black moves right; black jumps; white moves left. The solution requires 15 moves. Here is a table of the results:

number of black marbles	number of moves needed
1	3
2	8
3	15
4	24

If we use "$\square$" to represent the number of black marbles, and "$\triangle$" to represent the number of moves required, the function can be represented by this formula:

$$(\square \times \square) + (2 \times \square) = \triangle.$$

(iii) *The Tower of Hanoi*. This puzzle is also available from World Wide Games, Delaware, Ohio 43015.

I won't analyze it here, but if $\square$ represents the number of discs, and $\triangle$ the minimum number of moves required to solve the puzzle, then the function relating $\square$ and $\triangle$ can be represented (incompletely) by this table

□	△
1	1
2	3
3	7
4	15
⋮	⋮

or by the formula

$$2^{\square} - 1 = \triangle.$$

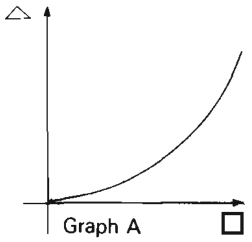
(iv) There are many functions in a child's environment that can be studied. Most will be too irregular for representation by a *formula*, but representation by a *graph* is always possible. Here are some examples:

a) If you drop a rubber ball, you can count how often it bounces. If you drop it from a greater height, it will tend to bounce more often. You can represent the relationship by a graph: measure the height from which the ball is dropped, and call that the $\square$ number; count how often the ball bounces, and call that the $\triangle$ number. Now make a *table*, which *might* look somewhat like this:

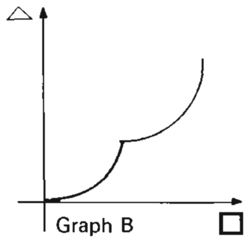
□	△
10 inches	2 bounces
12 inches	2 bounces
15 inches	4 bounces
18 inches	5 bounces
20 inches	5 bounces
25 inches	5 bounces

Finally, make a graph. Depending upon the kind of ball you use, this *may* turn out to be a surprise!

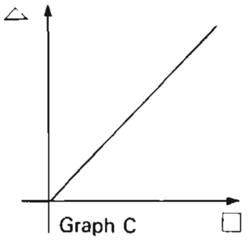
b) Alternatively, instead of counting the number of times the ball bounces, you could measure the rebound height of the first bounce; again dropping the ball from different heights, making a table, and making a graph.



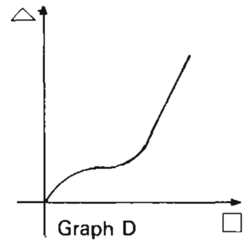
Graph A



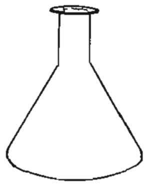
Graph B



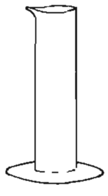
Graph C



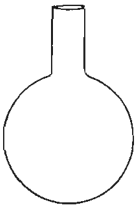
Graph D



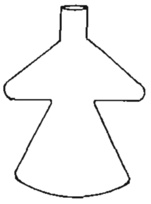
Bottle 1



Bottle 2



Bottle 3



Bottle 4

c) You can take bottles of different shapes, and fill them with water using a small paper cup or medium size dipper. For each different bottle, make a separate graph, with the number of dippers of water marked as the "☐" number on the horizontal axis, and the height of the water in the bottle marked as the "△" number on the vertical axis. After several graphs have been made for different bottles, mix them up, and see if students can tell *which graph* should go with which bottle. (You *can* tell, if you think about it awhile, and if the bottles are sufficiently different from one another.)

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SPECIAL NOTE TO TEACHERS

This is a "modernized" and "up-dated" version of mathematics materials that have been successfully used, and continually improved, for over twenty years. An earlier version of these materials, published by Addison-Wesley, used the format of a book for students (called the Student Discussion Guide), and an accompanying book for teachers. For the present version of these materials we have chosen to make available a single volume, combining the contents of the earlier Student Discussion Guide (which we print as the smaller left-hand column on most pages) with the text for teachers.

Opposite each exercise of the Student Discussion Guide, in the right-hand column, is the answer to the exercise, along with helpful comments and suggestions for teaching the material. In most chapters you will also find background material or introductory information preceding the Answers and Comments.

You have, therefore, in one convenient location, the Student Discussion Guide, the answers for every exercise, helpful teaching suggestions, and mathematical background material.

The key to using this material successfully is flexibility. Teachers need to adjust the presentation to the individual student or class. Please feel free to modify the order of topics, the pace, or the amount of review, etc., to achieve what seems best for *your* students.

chapter 1 / *Pages 1–2 of Student Discussion Guide*

STATEMENTS: TRUE, FALSE, AND OPEN

The material in the *Student Discussion Guide* provides the basis for experience with true, false, and open statements; substituting into equations; ordered pairs; and inequality symbols. The accompanying suggestions for the teacher relate, of course, to the same material and include, also, comments on the rule for substituting.

Before proceeding with Chapter 1 in detail, we make one mathematical remark on the modern use of language.

In the old days we used these names:

"$3 + \square = 5$" was called an *equation.*

"$3 < \square < 5$" was called an *inequality.*

"2" was called a *root* of the equation $3 + \square = 5$.

The symbol $\square$ or x was called a *variable* or an *unknown.*

"______ is sitting in the front row," was not used at all.

If you want to use the "modern" language of the Madison Project, here is how it goes:

"$3 + \square = 5$" is called an *open sentence.*

"{2}" is called the *truth set* for the open sentence $3 + \square = 5$ (because inserting 2 into the $\square$ makes the statement $3 + \square = 5$ become *true*).

"$3 < \square < 7$" is also called an *open sentence.*

If we consider only whole numbers, then: {4, 6, 5} is the *truth set* for the open sentence $3 < \square < 7$.

"______ is sitting in the front row" is called an *open sentence.*

The symbols $\square$ or x or ______ or △, etc., are called *placeholders.* (In the University of Illinois Committee on School Mathematics Project, the symbol $\square$ or x is called a *pronumeral*, by analogy with *pronoun.*)

A *set* may contain *one* element (for example, the set {2}), or it may contain *several elements* (for example, the set {2, 4, 6, 8}, which could be described as the set of even numbers less than 10), or it could contain a never-ending sequence of elements (for example, the set {1, 3, 5, 9, . . .}, that is, the set of *odd* whole numbers). Note that mathematicians use the final three dots at the end of a list to mean that the written list terminates, but the actual sequence we have in mind does not terminate; it goes on forever.

A set may contain *no* elements. (We shall write this "empty" set as {▨}; for example, the truth set for the open sentence $\square + 1 = \square$ is {▨}. As a second example, the truth set for the open sentence, "______ was President of the United States in 1066," is the empty set {▨}. That is to say, there is *no* proper name that can be substituted for the placeholder "____" in order to make this open sentence a true statement.

It is common to use braces in writing sets. Thus the set of even numbers could be written: {2 4, 6, 8, . . .}. Similarly, the set of integers from 1 to 10 inclusive could be written: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

A set is given as a *list;* the *order is not important*. The last set above could be written as: {2, 7, 3, 4, 1, 9, 8, 5, 6, 10}. Students often dictate sets in such an order. This is perfectly all right, it may even indicate more thought than if they merely rattled them off in the usual "counting" order.

Chapter 1

STATEMENTS: TRUE, FALSE, AND OPEN

[page 1]

What number can we write in the $\boxed{}$ to make a **true** statement?

(1) $3 + \boxed{} = 5$

(2) $8 + \boxed{} = 12$

(3) $5 + \boxed{} = 16$

ANSWERS AND COMMENTS

(1) {**2**}

(2) {**4**}

(3) {**11**}

The method by which the student will solve these three equations will be by "guessing" or "trial and error." You may choose to write the equation on the board, read it ("Three plus the number in the box equals five"), or simply ask, "What number shall I write in the box in order to make this a *true* statement?"

When a child suggests an answer, you may wish to try the following procedure:

Student: Three.

	Teacher writes on the board:
Teacher: All right, let's try three . . .	$3 + \boxed{3} = 5$
three plus three equals six . . .	$\underset{6}{3 + \boxed{3}} = 5$
and so this says six equals five.	$3 + \boxed{3} = 5$ $6 = 5$
Is that true or false?	

Student: False.

Teacher: All right, then we know that three doesn't work. What shall I try next?

A good question that you might ask here is: Was three *too large* or *too small?* Put a listing on the board:

Too small	Too large
	3

(Of course, 3 was too large because it made the left-hand side of the equation equal 6, whereas it *should* equal 5.) This sort of question gives you some control over the amount of random guessing.

The children should guess—and usually they guess very shrewdly. Nearly all of their guesses will reflect at least some insight into the nature of the problem. But if you think some of your children are guessing too wildly, you can use devices such as the "too small" and "too large" lists to impose a little more system into the children's thinking.*

After the children have solved one or two of these correctly, you may want to tell them that $3 + \square = 5$ is called an *open sentence* and $\{2\}$ is called the *truth set.*

After question 1 or question 2, you may want to insert an open sentence using words. For example:

Can you find the truth set for this open sentence?
______ is sitting in the front row.

Answer: The truth set, of course, will depend upon your class. It might look something like this:
{Marie, Nikki, Lex, Ellen, Cecile}.

Which of these are **true**? Which are **false**? Which are **open**?

(4) $\frac{1}{2} + \frac{1}{2} = 1$

(5) $3 \times 5 = 15$

(6) $39 + 21 = 50$

(7) $2 \times 2\frac{1}{2} = 5$

(4) **True**

(5) **True**

(6) **False**

(7) **True**

The point of these problems is to emphasize that all the statements can be classified under three headings: false, true, or open. Moreover, whenever a correct substitution is made for an *open* sentence, the resulting statement will be either true or false.†

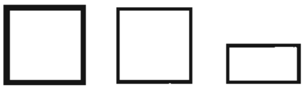
This is the first problem in this lesson that may elicit wrong answers from the children, most often because they will misread it as 2 *plus* $2\frac{1}{2} = 5$.

It is usually sufficient to point to the *multiplication* sign to put them back on the right track.

If you want to discuss the arithmetic here, especially with very young children, you can do so without recourse to any fancy rules for multiplying fractions or whatever. You can proceed *pictorially:*

This is a *thing.*

This is $2\frac{1}{2}$ *things.*



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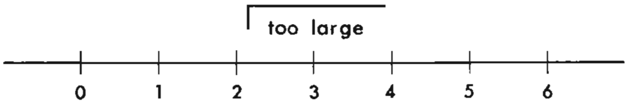
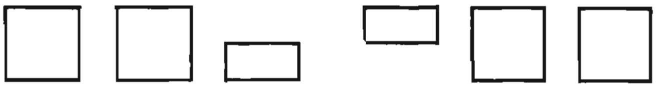
* For those of you who use the number line, here is a good opportunity for visualizing what is going on. You can mark the number line:



† If you are very resourceful you can see that this picture is oversimplified on several counts. While it is nice for those teaching this material to *know* all sorts of things, including all the pathological "exceptional cases," *explanations for the children should be as scanty as possible and should be restricted to the truly basic fundamentals.*

Fancy exposition has been avoided in this teachers' text, for fear it might convey the erroneous impression that all these things should be conveyed to the children. This is not the case. The teacher should tell the children very little.

Here we have $2\frac{1}{2}$ things and $2\frac{1}{2}$ *more* things:



How many things do we have?

This may be an instance of the Chinese motto that "a picture is worth a thousand words."

(8) $\frac{1}{2} + \frac{1}{3} = 2\frac{1}{4}$

(8) **False**

The idea here is to see if any of the children can answer immediately, without computing, on the grounds that $\frac{1}{2}$ and $\frac{1}{3}$ are *too small* to add together to produce a sum as large as $2\frac{1}{4}$.

This is a digression from the main theme of the lesson, and probably should not receive much stress. Indeed, it could be omitted with no great loss.

(9) $3 + \square = 9$

(9) **Open** The truth set would be {6}.

(10) Marie says that you can't call the statement

$$3 + \square = 9$$

"**true**," because you can't be sure what people will write in the $\square$.

What do you think?

(10) **Marie has a good point.**

(11) Would you call

$$3 + \square = 9$$

true, or **false**, or **open**?

(11) **Open**

(12) Can you make up an **open sentence**?

(12) **Your students will probably be able to do this readily enough.**

(13) Alan says that for the open sentence

$$8 + \square = 9,$$

the **truth set** is

{1}.

What do you think?

(13) **Alan is right.**

(14) Tony says that for this open sentence

$$5 + \square = 26,$$

the **truth set** is

{11}.

Do you agree?

(14) **No; the truth set is {21}.**

Can you find the truth set for each open sentence?

(15) $6 + \square = 10$

(16) $20 + \square = 50$

(17) $3 + (2 \times \square) = 11$

(15) {**4**}

(16) {**30**}

(17) {**4**}

Students usually find problems 15, 16, and 17 easy, and can answer them immediately. Do *not* explain the use of parentheses.

The following is a step-by-step description of what you might write on the board and say if a student suggests substituting 5 as the answer for problem 17.

Say:	Write on the board:
	$3 + \left(2 \times \boxed{}\right) = 11$
All right, let's try five.	$3 + \left(2 \times \boxed{5}\right) = 11$
Two times five is ten . . .	$3 + \left(2 \times \boxed{5}\right) = 11$ 10
and we have to add three . . .	$3 + \left(2 \times \boxed{5}\right) = 11$ $3 + 10$
three and ten are thirteen . . .	$3 + \left(2 \times \boxed{5}\right) = 11$ $3 + 10$ 13
which is not equal to eleven . . .	$3 + \left(2 \times \boxed{5}\right) = 11$ $3 + 10$ $13 \neq 11$

so five doesn't work.
Do you have any other suggestions?

The way you substitute 5 on the board is important. Try to do it with a casual air, realizing that the student is going to imitate your use of parentheses. (Do not tell him about the use of parentheses. He should observe for himself how they are used.)

Of course, the right answer to problem 17 is 4:

$$3 + \left(2 \times \boxed{4}\right) = 11$$

$$\begin{array}{c} 3 + \left(2 \times \boxed{4}\right) = 11 \\ 8 \end{array}$$

$$\begin{array}{c} 3 + \left(2 \times \boxed{4}\right) = 11 \\ 3 + 8 \\ 11 = 11 \end{array}$$

(18) $5 + \left(2 \times \boxed{}\right) = 25$

(19) $1 + \left(2 \times \boxed{}\right) = 17$

(20) $3 + \left(2 \times \boxed{}\right) = 203$ [page 2]

(21) $3 + \left(2 \times \boxed{}\right) = 8$

(18) **{10}**

(19) **{8}**

(20) **{100}**

(21) **{$2\frac{1}{2}$}**

Students usually find problems 18 through 21 easy, once they have seen the teacher work with the parentheses in problem 17.

Students sometimes say that there is no answer to problem 21. This is a clever remark, for the student who says this has discovered for himself the fact *that there is no whole number solution*. Ahh . . . but how about fractions?

When, as with problem 21, the first few guesses are not likely to work out, it is well to suggest that students keep two lists, one labeled "too small" and the other labeled "too large":

	Too small	Too large
$3 + \left(2 \times \boxed{}\right) = 8$	2	3
	1	4

You will want to make more use of these "too small" and "too large" tables in later lessons.

(22) ______ sits in the front row.

(22) **The answer depends on your class. A typical answer might look like this:**
{Lex, Peter, Debbie, Ellen, Sarah}.

The truth set will probably have several elements. If any one of them is substituted, the sentence becomes true; for example, <u>Ellen</u> sits in the front row.

(23) ______ teaches this class.

(23) **The answer should be written in set notation, using braces. For example, {Miss McQueen}.**

(24) ______ sits nearest to the door.

(24) **The answer depends on your class. A typical answer might be {Nikki}.**

(25) "____" is a country on the continent of (______).

(25) **The truth set here can be presented as a *table*:**

"__________"	(__________)
United States	**North America**
Ghana	**Africa**
Italy	**Europe**

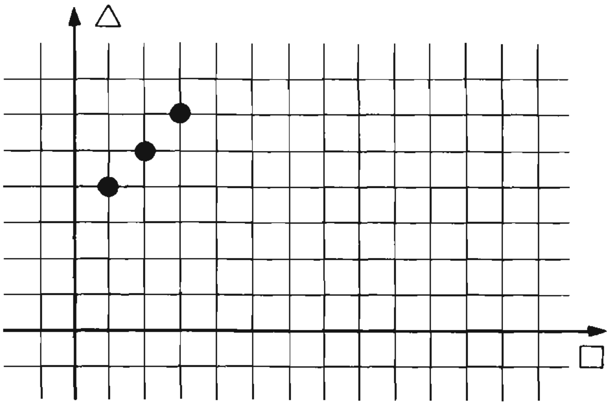
etc.

Table for Truth Set

The point of problem 25 is to prepare the way for problems such as △ = □ + 3.

□	△
1	4
2	5
3	6
⋮	⋮

Table for Truth Set



Graph for Truth Set

In either case, each element of the truth set is itself an *ordered pair*, such as, (Ghana, Africa) or else (1, 4).

In examples with boxes and triangles for placeholders, always give the number to go in the box *first*, and the number to go in the triangle *second*. For example, for the open sentence $\triangle = \square + 3$, the ordered pair (1, 4) would lead to the *true* statement $\underset{4}{\triangle} = \boxed{1} + 3$, whereas the ordered pair (4, 1) would lead to the *false* statement $\underset{1}{\triangle} = \boxed{4} + 3$. Consequently, (1, 4) belongs to the truth set, whereas (4, 1) does not.

(26) ______ is the first month of the year.	(26) **{January}**
(27) Do you know what this symbol means? $<$	(27) **The symbol $<$ means *is less than.***
(28) Joan says that $3 < 5$ is a true statement. What do you think?	(28) **Joan is right; it is a true statement.**
(29) Hal says that $5 < 3$ is a false statement. Do you agree?	(29) **Hal is right; it is a false statement.**
(30) Can you guess what the symbol $<$ means?	(30) **The symbol $<$ means *is less than* [or *is smaller than*, or *lies to the left* (*on the number line*) *of*].**
(31) Is this true or false? $8 < 21$	(31) **True**
(32) Can you make up a **false** statement using this symbol? $<$	(32) **Students usually have no trouble with this. Some right answers are: $5 < 3$; $100 < 50$; $2 < 1$; and so on.**
(33) Can you make up a **true** statement using this symbol? $<$	(33) **You can expect that the student will find this perfectly easy. A few possible answers: $1 < 2$; $3 < 5$; $1960 < 1961$; $1066 < 1732$; $0 < 1$; $5 < 33$; $1 < 1{,}000{,}000$; and so on.**
(34) Using only whole numbers, can you find the truth set for this open sentence? $3 < \square < 8$	(34) **{4, 7, 6, 5}**

Remember, the order of listing the elements of a *set* is unimportant. Mary and Jane are the same girls as Jane and Mary. The answer to question 34 could also be given {4, 5, 6, 7}, {7, 6, 5, 4}, {7, 4, 6, 5}, and so on.

(Of course, in making tables and graphs one deals with *ordered pairs*, and the order of listing *is* important: (3, 4) would make the sentence $\triangle = \square + 1$ become *true*, whereas (4, 3) would make it *false*, because of the convention that the *first* number goes into the placeholder box, and the *second* number goes into the placeholder triangle.)

It is better not to tell *the above to the students. It is included here only for your information. It is important for you to* understand, *and for the student to* discover.

(35) Lex says that there is something called the **rule for substituting**. Do you know what it is?

(35) The rule for substituting states: If one open sentence contains several placeholders *of the same shape* (say, for example, a box), then *whatever number* you put in the *first* of these placeholders, *you must put this same number in all others.*

Some examples will show what this rule *does* mean, and what it *does not* mean.

(a) In the open sentence $\square \times \square = 16$, the rule for substituting says that, if 3 is put in the first box, $\boxed{3} \times \square = 16$, then 3 must be put in all the other boxes, $\boxed{3} \times \boxed{3} = 16$. (By now you have probably noticed that whenever numbers are substituted into an open sentence, it is changed into a *new* statement that is no longer open, but will be either *true* or else *false*. For example, for the open sentence $\square \times \square = 16$, if we substitute 3 into every box, we get the *false* statement $3 \times 3 = 16$.)

(b) In the open sentence $(\square \times \square) - (5 \times \square) + 6 = 0$, if 0 is substituted into the first box,

$$(\boxed{0} \times \square) - (5 \times \square) + 6 = 0,$$

then the rule for substituting says that 0 must be substituted into all the other boxes,

$$(\boxed{0} \times \boxed{0}) - (5 \times \boxed{0}) + 6 = 0$$

or

$$(0 \times 0) - (5 \times 0) + 6 = 0$$

or

$$0 - 0 + 6 = 0,$$

(which, of course, is a false statement).

(c) In the open sentence $\square + 0 = \square$, if 3 is substituted into one box,

$$\boxed{3} + 0 = \square,$$

then according to the rule for substitution 3 must be substituted into the other box

$$\boxed{3} + 0 = \boxed{3}$$

(which, of course, is a true statement).

(d) In later lessons, placeholders of *different shapes* will be used, for example, boxes and triangles.

Whatever number you put in one box must be put in all the other boxes.

Whatever number you put in one triangle must be put in all the other triangles.

The number in the box may be different from the number in the triangle, or they may be the same. That is up to you.

In the open sentence

$$\square + \triangle = \triangle + \square,$$

these substitutions *are correct* according to the rule for substituting:

$$\boxed{7} + \overset{\triangle}{3} = \overset{\triangle}{3} + \boxed{7}$$

$$\boxed{0} + \overset{\triangle}{1} = \overset{\triangle}{1} + \boxed{0}$$

$$\boxed{4} + \overset{\triangle}{4} = \overset{\triangle}{4} + \boxed{4}$$

These substitutions *are not correct*, according to the rule for substituting:

$$\boxed{3} + \overset{\triangle}{1} = \overset{\triangle}{3} + \boxed{1}$$

$$\boxed{5} + \overset{\triangle}{2} = \overset{\triangle}{2} + \boxed{7}$$

(Remember, you check correctness according to the rule for substituting by asking:

$$\boxed{8} + \overset{\triangle}{1} = \overset{\triangle}{2} + \boxed{8}$$

Is the number in one box also in all the other boxes?

Answer: Yes, correct.

$$\boxed{8} + \overset{\triangle}{1} = \overset{\triangle}{2} + \boxed{8}$$

Is the number in one triangle also in all the other triangles?

Answer: No, incorrect.

This substitution is *not* correct according to the rule for substitution.)

(e) Notice that correctness according to the rule for substitution is not the same idea as whether the resulting statement is *true* or *false:*

For the open sentence $\square \times \square = 16$, the substitution $\boxed{3} \times \boxed{3} = 16$ is *correct* according to the rule for substituting, but the statement $3 \times 3 = 16$ is *false*.

The substitution $\boxed{2} \times \boxed{8} = 16$ is *wrong* according to the rule for substituting, but the statement $2 \times 8 = 16$ is *true*.

Of course, in looking for the *truth set* you want to "win this both ways." You seek substitutions that are correct according to the rule for substituting and that also produce *true* statements. For example:

$$\boxed{4} \times \boxed{4} = 16.$$

It seems advisable not to tell all of this to the children in answer to question 35. Try to deal with most of these ideas in subsequent lessons, whenever they arise naturally.

(36) For this open sentence,

$$\boxed{} \times \boxed{} = 16,$$

can you substitute **correctly** according to the rule for substituting so that you get a **false** statement?

(36) There are many possible answers. Here are a few:

$$\boxed{3} \times \boxed{3} = 16$$

$$\boxed{7} \times \boxed{7} = 16$$

$$\boxed{8} \times \boxed{8} = 16$$

$$\boxed{1961} \times \boxed{1961} = 16$$

$$\boxed{0} \times \boxed{0} = 16$$

and so on . . .

(37) For this same open sentence, can you make a **wrong** substitution according to the rule for substituting so that you will get a **true** statement?

(37) Most children give this answer, which is a good one,

$$\boxed{2} \times \boxed{8} = 16.$$

This substitution is *wrong* according to the rule, but the statement

$$2 \times 8 = 16$$

is *true*. *This is what was requested.* Other correct answers include:

$$\boxed{\tfrac{1}{2}} \times \boxed{32} = 16$$

$$\boxed{0.1} \times \boxed{160} = 16$$

$$\boxed{1.6} \times \boxed{10} = 16$$

etc.

(38) Now, can you substitute **correctly** according to the rule for substituting so that you get a **true** statement?

(38) $\boxed{4} \times \boxed{4} = 16$

This, of course, is the usual or "proper" question that determines the truth set. Here you "win both ways": The substitution complies with the rule for substituting. The statement $4 \times 4 = 16$ is true.

This leads directly to question 39.

(39) What is the truth set for this open sentence?

$$\boxed{} \times \boxed{} = 16$$

(39) $\{4\}$

When working with *signed* numbers, this truth set is $\{^{+}4, {}^{-}4\}$. The children do not yet know this, and they rarely suggest such a possibility at this point. Do not suggest such a possibility. The matter will come up a few lessons later, and by then the children will be ready to deal with it.

chapter 2 / *Pages 3–4 of Student Discussion Guide*

CAN YOU ADD AND SUBTRACT?

This entire chapter is ordinarily part of the first 45-minute lesson. The "pet store example" should take about two or three minutes. Children and teachers usually enjoy this immensely; especially if their pet store sells dogs, cats, trained rattlesnakes, horses, and quite a few other items—some quite reasonable and some not very. (The pet store sequence is available on one of the tape-recorded classroom lessons and in the film entitled, "First Lesson." An alternate approach, using pebbles in a bag, is presented in the film, "A Lesson with Second Graders.")

Chapter 2
CAN YOU ADD AND SUBTRACT?

[page 3]

ANSWERS AND COMMENTS

Suppose we are operating a pet store, and we come into the store on Monday morning. We unlock the door, unlock the cash register (which has quite a lot of money in it), and feed the animals.

Now, somebody comes in to buy something.

The answers to the pet-store problems depend on whatever the students suggest or on answers to the previous problems in this series. Hypothetical answers have been given to problems 1 through 15 to show how the sequence might go.

(1) What does he buy?

(1) **Dog**

(2) How much does he pay us for it?

(2) **$5** Teacher writes on board: 5

As a matter of prudence, you might find it advisable to round off the students' answers to an even number of dollars. For example, if they say "$4.50," you might say, "Well, let's call it $5. I don't want to get into *hard* numbers." They will think this is a good joke.

(3) Is there now **more** money in the cash register than there was when we opened up this morning, or is there **less**?

(3) **More**

This is a "stupid" question. Children usually find its naiveté a source of fascination. As a child psychiatrist has remarked: In fairy tales, the giants are always stupid. I guess this is just the way that grownups look in the eyes of a child.

(4) How much less, or how much more?

(4) **$5 more**

(5) Now somebody comes in to return something. What does he return?

(5) **Parakeet**

(6) How much money do we give back?

(6) **$25** Teacher writes on board: $5 - 25$.

(7) Is there now more money in the cash register than there was when we first opened up this morning, or is there less?

(7) **Less**

(8) Do you know how much more, or how much less?

(8) **$20 less** Teacher writes on board: $5 - 25 = {}^{-}20$

(9) Do you know how to write this?

(9) $^{-}20$ (which we read "negative twenty")

Do *not* write a big sign centered, like this:

$$-20.$$

Write a *small* sign, above the center:

$$^{-}20.$$

This notation works better in subsequent lessons where the arithmetic of signed numbers is developed, since it is clear that $^{-}2 \times {}^{+}3$ is actually a multiplication problem.

Students generally have difficulty determining what kind of problem is meant by the old-fashioned

$$-2 \times +3$$

(and you cannot especially blame them). But with the new notation any difficulty here has completely disappeared.

In a similar way, read $^{+}2$ as a "positive two," and *not* as "plus two." The number $^{-}3$ can be read as "negative three," but do *not* read it as "minus three." Of course, hopefully the children will imitate your usage, but do not jump on them terribly hard if they say "minus" or "plus" where you would have said "negative" or "positive."

The point here is to distinguish between the *sign* of a number

$^{+}2$ positive two
$^{-}3$ negative three

vs. an *operation*

$2 + 3$ add
$2 - 3$ subtract.

This distinction is important. Observing it carefully will make many of the following lessons much easier.

(10) But, it's not yet noon. Now somebody else comes in to the store and buys a ______.

(10) Donkey

Teacher writes on the board:

$5 - 25 + 50 =$

(11) How much does he pay?

(11) $50

(12) Is there now more money in the cash register than there was when we first opened up this morning, or is there less?

(12) More

(13) How much more, or how much less?

(13) $30 more

(14) How can we write this?

(14) $5 - 25 + 50 = {}^{+}30$

$5 - 25 + 50 = {}^{+}30$

(15) Now it is noontime, and we close the shop and go home for lunch. Have we made money or lost money this morning?

(15) We've made money. We have more money in the cash register than we had when we opened up this morning, namely, $30 more.

(16) Can you read each of these numbers?

$^{+}3$
$^{-}10$
$^{+}500$

(16) $^{+}3$ is read "positive three"
$^{-}10$ is read "negative ten"
$^{+}500$ is read "positive five hundred"

(17) Now it is Monday morning, one week later. We come into the store, unlock the cash register (which has quite a lot of money in it), and feed the animals. Somebody comes into the store and buys a ________. [page 4]	(17) through (30) **This sequence parallels the sequence of problems 1 through 15.**
(18) How much does he pay?	
(19) Are we richer or poorer?	
(20) By how much?	
(21) Now, a man comes in and returns a horse. He says he wants his money back because the horse is too big for his apartment. (He has a rather small apartment.) How much do we give back to him?	
(22) Is there more money now than when we first opened this morning, or is there less?	
(23) How much more, or how much less?	
(24) How can we write this?	
(25) But, the morning isn't over yet. A man comes in and buys a ________.	
(26) How much does he pay?	
(27) Is there more money now than when we opened up this morning, or is there less?	
(28) How much more, or how much less?	
(29) How can we write this?	
(30) Have we made money this morning, or have we lost money? How much? Now we lock up the shop and go home to lunch.	
(31) Can you read these numbers? $^{+}8$ $^{-}21$ $^{+}2$	(31) **$^{+}8$ is read ''positive eight''** **$^{-}21$ is read ''negative twenty-one''** **$^{+}2$ is read ''positive two''**
(32) Which would you rather earn, $^{+}5$ dollars or $^{-}5$ dollars?	(32) **$^{+}5$ dollars, because then we would be *richer* by five dollars; if we earned $^{-}5$ dollars, we would be *poorer* by five dollars.**
(33) Which is more, $^{+}0$ or $^{-}0$?	(33) **They are equal. If we don't lose any money and don't win any money, it doesn't matter whether we *won* zero dollars, or *lost* zero dollars.**
(34) Joan says that 2 means the same as $^{+}2$. Do you agree?	(34) **Yes. If we write no sign on a number, that means the same thing as if we had written a *positive* sign; 2 is the same as $^{+}2$.**

chapter 3 / *Pages 5–6 of Student Discussion Guide*
SOME OPEN SENTENCES

Chapter 3
SOME OPEN SENTENCES

[page 5]

How good are you at discovering things in science and mathematics?

There is a secret way to solve the problems in this chapter.

If you discover the secret method . . .

PLEASE
DON'T
TELL
!

(It's a secret.)

(1) Can you find the truth set for this open sentence?

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$\{\ \ ,\ \ \}$

ANSWERS AND COMMENTS

(1) $\{\mathbf{2, 3}\}$*

To solve this problem the student must proceed by trial and error. One method of leading them to the solution is to ask them to guess a number; then substitute the number they choose, to see if it will work. Here is a possible sequence (note close parallel to pet store problem of Chapter 2).

Teacher writes on the board:

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

Teacher: Who's good at guessing?"

Student: Four.

Teacher: All right, let's try four. If I put a four in this box, what must I do now?

$$(\boxed{4} \times \square) - (5 \times \square) + 6 = 0$$

Student: Write four in all the other boxes.

Teacher: All right, now, let's see. A man comes in and buys something for $16 . . . At this point, are we *richer* or *poorer?*

$$(\boxed{4} \times \boxed{4}) - (5 \times \boxed{4}) + 6 = 0$$

$$\overset{16}{(\boxed{4} \times \boxed{4})} - (5 \times \boxed{4}) + 6 = 0$$

Student: Richer by $16.

---

* Remember, this could also be written {3, 2}. For a *set,* the order is unimportant; for an *ordered pair* (to be studied in later lessons), the order is important.

Teacher writes on the board:

Teacher: . . . *and then* somebody comes in, and we *give him* back \$20 . . . At this point, are we *richer* or *poorer?*

$$\begin{array}{ccccc} 16 & - & 20 & & \\ (\boxed{4}\times\boxed{4}) & - & (5\times\boxed{4}) & +\ 6 & = 0 \end{array}$$

Student: Poorer by \$4.

$$\begin{array}{ccccc} 16 & - & 20 & = {}^{-}4 & \\ (\boxed{4}\times\boxed{4}) & - & (5\times\boxed{4}) & +\ 6 & = 0 \end{array}$$

Teacher: All right . . . but it isn't yet noontime.

(erasing "= ${}^{-}4$")

Now somebody comes in and spends \$6 . . .

$$\begin{array}{ccccc} 16 & - & 20 & +\ 6 & = \\ (\boxed{4}\times\boxed{4}) & - & (5\times\boxed{4}) & +\ 6 & = 0 \end{array}$$

Now, are we *richer* or *poorer* than we were when we *started* this morning?

Student: Richer by \$2.

Teacher: All right, then how much is

$$16 - 20 + 6?$$

Student: Positive two.

Teacher: All right . . .

$$\begin{array}{ccccc} 16 & - & 20 & +\ 6 & = {}^{+}2 \\ (\boxed{4}\times\boxed{4}) & - & (5\times\boxed{4}) & +\ 6 & = 0 \end{array}$$

so, when we say that $16 - 20 + 6$ equals zero . . . is that *true* or *false?*

$$\begin{array}{ccccc} 16 & - & 20 & +\ 6 & = 0 \\ (\boxed{4}\times\boxed{4}) & - & (5\times\boxed{4}) & +\ 6 & = 0 \end{array}$$

Student: False.

Teacher: All right . . . so we know that four doesn't work. What other number should we try now?

$$(\boxed{}\times\boxed{}) - (5\times\boxed{}) + 6 = 0$$

This sequence repeats until the children find that two and three both work.*

A tape recording or sound film of this activity is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

(2) Have you discovered the secret? Remember — don't tell!

(2) Don't tell the students, but the secret is that the numbers which make the statement *true* can be found from the facts that:

$$\mathbf{2+3=5} \longrightarrow (\square\times\square)-(\mathbf{5}\times\square)+\mathbf{6}=0 \longleftarrow \mathbf{2\times 3=6}$$

$$\{\mathbf{2, 3}\}.$$

(3) Can you find the truth set for this open sentence?

$$(\square\times\square)-(8\times\square)+15=0$$

{ , }

(3) **{3, 5}**

The discussion may follow the same pattern as in question 1, or you may be able to shorten it somewhat by now.

Notice how the "secret" again works:

$$3+5=8 \longrightarrow (\square\times\square)-(8\times\square)+15=0 \longleftarrow 3\times 5=15$$

$$\{3, 5\}.$$

The main thing here is the light touch. It is usually sufficient to spend only 10 or 20 minutes on Chapter 3 during this lesson. This material is returned to frequently, and casually, in subsequent lessons.

This is usually one of our most exciting and gratifying lessons.

All students quickly learn how to substitute and test for roots by trial and error.

Don't tell the students, but here are the two "secrets":

$$(\square\times\square)-(13\times\square)+22=0$$

$$\{2, 11\}$$

$$2+11=13 \nearrow$$

$$2\times 11=22 \nearrow$$

Several students usually discover both of the two "secrets" (that the product of the roots is the last coefficient in the equation and the sum of the roots is the second coefficient). When they do, they can use their discovery to solve future equations faster than they can be written on the board. Encourage the students to protect their "secret," and not let the other students find out how they are doing it.

To make the "secret" work easily, most equations have roots that are prime. This includes questions 1, 3, 5, 12, and 16. Those equations with roots that are not prime are somewhat more difficult (as in the case of questions 4, 6, 9, 10, 11, 15, and 17), or even quite a bit more difficult (question 18).

Can you find the truth sets for these open sentences?

(4) $(\square\times\square)-(15\times\square)+50=0$

{ , }

(4) **{5, 10}**

(5) $(\square \times \square) - (13 \times \square) + 22 = 0$

$\{\ ,\ \}$

(5) **{11, 2}**

(6) $(\square \times \square) - (102 \times \square) + 200 = 0$

$\{\ ,\ \}$

(6) **{2, 100}**

(7) Have you discovered the secret? Remember — don't tell!

(7) **Repeat of question 2.**

(8) How many secrets are there?

(8) **Two**

The purpose of this question is to prod any students who have found only *one* of the two rules and who think there are not any more. But don't give away hints as to what the secrets are.

Can you solve these equations?

(9) $(\square \times \square) - (17 \times \square) + 70 = 0$

$\{\ ,\ \}$

(9) **{10, 7}**

(10) $(\square \times \square) - (37 \times \square) + 70 = 0$

$\{\ ,\ \}$

(10) **{35, 2}**

A child who knows *only* the product rule might answer {10, 7} (which is wrong); a child who knows only the *sum* rule might answer {30, 7} (which is also wrong). Only {35, 2} satisfies *both* rules. The correct answer must always satisfy *both* rules.

(11) $(\square \times \square) - (1008 \times \square) + 8000 = 0$

$\{\ ,\ \}$

(11) **{8, 1000}**

[page 6]

(12) $(\square \times \square) - (7 \times \square) + 10 = 0$

$\{\ ,\ \}$

(12) **{5, 2}**

(13) Jerry says there is only one secret. Do you agree?

(13) **No; there are two secrets.**

The two secrets are the *sum* rule and the *product* rule, but do not give any hint as to what the secrets are. It is not advisable to call them *sum* rule and *product* rule within earshot of the children—these names are too suggestive and would give away the secret.

(14) Marie says there are two secrets. Do you agree?

(14) **Yes**

Compare the answers to questions 2, 3, 8, and 10 immediately above.

Can you find the truth set for each of these equations?

(15) $(\square \times \square) - (28 \times \square) + 75 = 0$

$\{\ ,\ \}$

(15) **{25, 3}**

(16) $(\square \times \square) - (16 \times \square) + 55 = 0$

$\{\ ,\ \}$

(16) **{5, 11}**

(17) $(\square \times \square) - (107 \times \square) + 700 = 0$

$\{\ ,\ \}$

(17) **{7, 100}**

(18) $(\square \times \square) - (20 \times \square) + 96 = 0$

$\{\ ,\ \}$

(18) **{8, 12}**

This is a very hard problem; the difficulty of one of these problems increases as we get more factorizations of the *constant* term in the equation. If it is a prime, for example,

$$(\square \times \square) - (12 \times \square) + 11 = 0,$$

11 is a prime

then the only possible roots are itself and one:

$$(\square \times \square) - (12 \times \square) + 11 = 0 \qquad \{11, 1\}.$$

If it is a product of two primes, then there are two possible factorizations:

$$(\square \times \square) - (7 \times \square) + 10 = 0 \qquad \{2, 5\}$$

or else

$$(\square \times \square) - (11 \times \square) + 10 = 0 \qquad \{10, 1\}.$$

If, however, there are many factorizations, then the problem becomes very much harder:

$$(\square \times \square) - (20 \times \square) + 96 = 0 \qquad \{\ ,\ \}$$

$$96 = 2^5 \times 3 = 2 \times 2 \times 2 \times 2 \times 2 \times 3.$$

We must find two factors whose sum is 20; to do this systematically, we might arrange them in order of size:

1\. 96 $1 \times 96 = 96$ $1 + 96 = 97 \neq 20$ (does not work)
2\. 48 $2 \times 48 = 96$ $2 + 48 = 50 \neq 20$ (does not work)
3\. 32 $3 \times 32 = 96$ $3 + 32 = 35 \neq 20$ (does not work)
4\. 24 $4 \times 24 = 96$ $4 + 24 = 28 \neq 20$ (does not work)
6\. 16 $6 \times 16 = 96$ $6 + 16 = 22 \neq 20$ (does not work)
8\. 12 $8 \times 12 = 96$ $8 + 12 = 20$ (Hooray!)

$$(\square \times \square) - (20 \times \square) + 96 = 0 \qquad \{8, 12\}$$

It's best not to tell the children how to do this. In a later lesson you might possibly supply a little guidance; however, in this first lesson it should be entirely up to them. If they never do find the answer (although they just about always do find it), well . . . that's life.

[For those of you who have extensive training in mathematics, we repeat at this point that the main purpose of this lesson is to give the children experience with the mathematical concept of *variable* (or "pronumeral"). It is *not* a lesson in quadratic equations.]

chapter 4 / *Pages 6–7 of Student Discussion Guide*

THE MATRIX GAME*

This is ordinarily part of the first 45-minute lesson. It need not be, however.

Chapter 4
THE MATRIX GAME

[page 6]

(1) There is a kind of mathematics known as **game theory**. Do you know where it is used?

We can play a **matrix game** ourselves. We can call one team the positive team. If the positive team is ahead three points, we will write the score as

$$\boxed{{}^{+}3}$$

The matrix game uses only *one* number to give the score (not *two*, as in baseball, where you might say "7 to 4").

We will call the other team the negative team.

(2) What would we mean by this score?

$$\boxed{{}^{-}2}$$

To play, we need a matrix:

$$\begin{array}{cc} & \begin{array}{ccc} W & X & H \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} {}^{+}2 & {}^{-}1 & {}^{-}1 \\ {}^{-}1 & 0 & {}^{+}1 \\ {}^{-}1 & {}^{+}1 & 0 \end{pmatrix} \end{array}$$

The players on the positive team will choose either *A*, or *B*, or *C*. Players on the negative team will choose either *W*, or *X*, or *H*.

[page 7]

We need two assistants in this game. The assistants watch to see what each player writes, and *after both* players have chosen a letter, the assistants announce the choices.

The best way to see how the game works is to watch while two teams play.

Starting score: $\boxed{0}$

First play: When *both* teams have written their choices, the "assistants" can report:

The positive team wrote *C*.
The negative team wrote *X*.

By writing *C*, the positive team chose the third row:

$$\begin{array}{cc} & \begin{array}{ccc} W & X & H \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} {}^{+}2 & {}^{-}1 & {}^{-}1 \\ {}^{-}1 & 0 & {}^{+}1 \\ {}^{-}1 & {}^{+}1 & 0 \end{pmatrix} \end{array}$$

ANSWERS AND COMMENTS

(1) **"Game theory" is used in making certain kinds of decisions in military situations, in industry, commerce, and so on.**

A large amount of literature on this subject has appeared since World War II. You might, for example, enjoy looking at some issues of the *Journal of the Operations Research Society of America*.

(2) **A score of $^{-}2$ (negative two) would mean that *the negative team is two points ahead.***

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*A tape recording or sound film of this activity is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

By writing *X*, the negative team chose the second column:

$$\begin{array}{c|ccc} & W & \boxed{X} & H \\ \hline A & ^{+}2 & ^{-}1 & ^{-}1 \\ B & ^{-}1 & 0 & ^{+}1 \\ \boxed{C} & ^{-}1 & ^{+}1 & 0 \end{array}$$

(3) What number is in the *C* row and also in the *X* column?	(3) **$^{+}1$ (positive one)**
(4) What is the score now?	(4) **$^{+}1$ (The positive team is ahead by one point.)**
Second play: The positive team wrote *A*. The negative team wrote *W*. (5) What is the score now?	(5) **$^{+}3$ (The positive team was one point ahead; it just *won* two points more; it is, therefore, now three points ahead, which we indicate by saying that the score is positive three.)**
Third play: The positive team wrote *A*. The negative team wrote *X*.	
(6) What is the score now?	(6) **$^{+}2$ (The negative team just won one point, so the positive team is now ahead by only 2 points.)**
Why don't you play?	
(7) Have you played a game yourself? (8) Who won?	(7) and (8) **Rhetorical questions**
(9) What was the final score?	(9) **This, of course, depends on your class.**
(10) Can you find the truth set for this open sentence? $^{+}5 + \square = {}^{+}3$ $\{\quad\}$	(10) **$\{^{-}2\}$ (negative two)**

chapter 5 / *Pages 8–9 of Student Discussion Guide*

IDENTITIES

Chapter 5
IDENTITIES

[page 8]

Which are **true?**
Which are **false?**
Which are **open?**

(1) $2 \times 2\frac{1}{2} = 5$

(2) $29 + 51 = 70$

(3) $8 \times 8 = 16$

(4) $\frac{1}{4} + \frac{1}{5} = 1\frac{1}{4}$

(5) $2 \times 1\frac{1}{2} = 3$

(6) $\frac{7 \times 10}{7} = 10$

(7) $\frac{3 \times 2}{7} = 5$

(8) $2 \times 3\frac{1}{2} = 7$

(9) $5 + \square = 6$

(10) $12 + \square = 12$

(11) $^{+}8 + \square = {}^{+}6$

(12) $3 + (2 \times \square) = 6$

Can you find the truth set for each open sentence?

(13) $8 + \square = 12$ { }

(14) $6 + \square = 6$ { }

(15) $3 + (2 \times \square) = 8$ { }

(16) $2 + (3 \times \square) = 6$ { }

ANSWERS AND COMMENTS

(1) **True**

(2) **False**

(3) **False**

(4) **False**

The fractions $\frac{1}{4}$ and $\frac{1}{5}$ are clearly too small to add up to $1\frac{1}{4}$; what is being tested here is whether the children have developed any idea of the *size* of a number like $\frac{1}{4}$, $\frac{1}{5}$, or $1\frac{1}{4}$.

(5) **True**

(6) **True** It should not be necessary for the children to
(7) **False** multiply out the numerators; hopefully, they will soon learn to recognize that multiplying by 7 and then dividing by 7 gets you back where you started. (Mathematicians express this by saying that *multiplying by 7* and *dividing by 7* are *inverse operations.*)

(8) **True**

(9) **Open; the truth set is** $\{\mathbf{1}\}$.

(10) **Open; the truth set is** $\{\mathbf{0}\}$.

(11) **Open; the truth set is** $\{^{-}\mathbf{2}\}$.

(12) **Open; the truth set is** $\{\mathbf{1\frac{1}{2}}\}$.

(13) $\{\mathbf{4}\}$

(14) $\{\mathbf{0}\}$

(15) $\{\mathbf{2\frac{1}{2}}\}$

(16) $\{\mathbf{1\frac{1}{3}}\}$

It may take some time, using the "too large" and "too small" lists, before anybody solves this; a "number-line" picture may help.

(17) $(\square \times \square) - (7 \times \square) + 10 = 0$ { , }	(17) **{5, 2}**
(18) $(\square \times \square) - (12 \times \square) + 11 = 0$ { , }	(18) **{11, 1}**
(19) $(\square \times \square) - (3 \times \square) + 2 = 0$ { , }	(19) **{1, 2}**
(20) $(\square \times \square) - (17 \times \square) + 30 = 0$ { , }	(20) **{15, 2}**
[page 9]	
(21) $(\square \times \square) - (11 \times \square) + 30 = 0$ { , }	(21) **{5, 6}**
(22) $(\square \times \square) - (13 \times \square) + 30 = 0$ { , }	(22) **{10, 3}**
(23) $(\square \times \square) - (31 \times \square) + 30 = 0$ { , }	(23) **{30, 1}**
(24) $(\square \times \square) - (107 \times \square) + 700 = 0$ { , }	(24) **{7, 100}**
(25) $(\square \times \square) - (29 \times \square) + 100 = 0$ { , }	(25) **{25, 4}**
(26) $(\square \times \square) - (5 \times \square) + 6 = 0$ { , }	(26) **{2, 3}**
(27) $(\square \times \square) - (15 \times \square) + 26 = 0$ { , }	(27) **{13, 2}**
(28) Can you find a number that will make this open sentence true? $5 + \square = 7$	(28) **{2}**
(29) Can you find a number that will make this open sentence false? $5 + \square = 7$	(29) **There are many possible answers. To list a few: 3; 1; 0; $^{-}1$; 1960; 1,000,000; etc.**
(30) Can you make up an open sentence that will be true for every substitution?	(30) **There are many answers. Usually, the first answer children think of (perhaps after arguing for some time over whether or not there really *are* any) is:** $0 \times \square = 0$ **or else** $\square \times 0 = 0.$

Any open sentence that becomes true for every correctly made substitution is called an *identity.*
Examples:

(a) $3 + \square = 5$ is *not* an identity, because substituting 3 into the box would produce a false statement.

(b) $\square + 0 = \square$ *is* an identity (remember the *rule for substituting*).

(c) $3 \times \square = \square \times 3$ *is* an identity; *whatever* number is put into $\square$, the resulting statement will be true.

(d) $\square \times (\square + 1) = (\square \times \square) + \square$ *is* an identity, although most children do not think so at first.

(31) Jerry says this will be true for every substitution:

$$1 \times \square = 1.$$

Do you agree?

(31) No

Jerry had a good idea, but it did not quite work. You can prove that it is *not* an identity by substituting (say) 7 into the box; the resulting statement, $1 \times 7 = 1$, is *false*.

Hopefully, one of your children will eagerly volunteer to "fix this up" so that it *does* work. You can do this by changing the 1 on the right-hand side to a box:

$1 \times \square = \square$ *is* an identity.

(32) Al says that this is an **identity:**

$$0 \times \square = \square.$$

Do you agree?

(32) No

This involves the first use of the word "identity" with the children. You may want to give them the definition (compare the answer to question 30 above). Perhaps one of your children will get an idea on how to fix this up so that it *will* be an identity.

(33) Jay says that this is an **identity:**

$$\square + 1 = \square.$$

Do you agree?

(33) No

You can prove that $\square + 1 = \square$ is *not* an identity by substituting 10 into the box. Using 0 in place of the 1 will make this an identity:

$\square + 0 = \square$ *is* an identity.

(34) How many identities can you make up?

(34) Here are some identities. There are many more.

$(\mathbf{0} \times \square) + \square = \square$

$\square \times \mathbf{1} = \square$

$\mathbf{1} \times \square = \square$

$\square + \square = \mathbf{2} \times \square$

$\square + \square + \square = \mathbf{3} \times \square$

$\mathbf{2} \times \square = \square \times \mathbf{2}$

$\mathbf{3} \times \square = \square \times \mathbf{3}$

$\mathbf{1961} \times \square = \square \times \mathbf{1961}$

$\mathbf{1732} + \square = \square + \mathbf{1732}$

$\mathbf{2} + \square = \square + \mathbf{2}$

$\mathbf{1066} + \square = \square + \mathbf{1066}$

$\mathbf{1685} + \square = \square + \mathbf{1685}$

$(\mathbf{1} \times \square) + (\mathbf{1} \times \square) = \mathbf{2} \times \square$

$(\mathbf{4} \times \square) + (\mathbf{5} \times \square) = \mathbf{9} \times \square$

$$\Box - \Box = 0$$

$$0 + \Box = \Box$$

$$\Box = \Box$$

$$\Box \times (\Box + 7) = (\Box \times \Box) + (\Box \times 7)$$

$$(\Box + 3) \times (\Box - 3) = (\Box \times \Box) - 9$$

$$(0 \times \Box) + 7 = 7$$

$$(\Box + 3) \times (\Box + 3) = (\Box \times \Box) + (6 \times \Box) + 9$$

$$(5 \times \Box) + (0 \times \Box) = (\Box \times 5)$$

$$(\Box + 0) \times 0 = 0$$

$$(\Box \times 0) + 0 = 0$$

(35) What do we mean by the **rule for substituting?**

(35) The *rule for substituting* says that, if an open sentence contains several boxes, then *whatever number is put into one box must be substituted into all the other boxes.*

(36) Can you make up an example that will violate the **rule for substituting?**

(36) Of course, there are many possible answers to this question. For example, in the open sentence $\Box \times \Box = 36$, each of the following substitutions violates the rule for substituting:

$$\boxed{7} + \boxed{8} = 36$$

$$\boxed{^{+}1} \times \boxed{^{-}1} = 36$$

$$\boxed{3} \times \boxed{12} = 36.$$

Notice that in the first two cases, the resulting statement is *false*, whereas in the last case, the resulting statement is *true*. The rule for substituting is *not* concerned with the truth or falsity of the resulting statement, but only with whether the *same* number is put into every box.

(37) What is the truth set for

$$\Box \times \Box = 49?$$

(37) $\{7\}$

To find the truth set for the open sentence $\Box \times \Box = 49$, we must, of course, find a substitution that wins both ways, i.e., a substitution that is correct according to the rule for substituting and, at the same time, produces a *true* statement. Among positive numbers, the only suitable substitution is 7, and so we say that the truth set is $\{7\}$.

Of course, in later lessons, we shall say that the truth set is $\{^{+}7, {}^{-}7\}$, but it would be premature to mention this now, for most classes.

(38) What do we mean by an **identity?**

(38) An identity is an open sentence that becomes true for every substitution, provided that the substitutions are made correctly (the rule for substituting is obeyed).

chapter 6 / *Page 10 of Student Discussion Guide*

Identities and Open Sentences

This chapter is concerned with the following problem: if someone shows you an open sentence, such as

$$(\square \times \square) + 6 = (5 \times \square),$$

how can you decide whether or not this is an *identity?*

The answer is that you cannot always decide with complete certainty.

An open sentence is an identity if *every* substitution produces a true statement. Consequently, if there is a single substitution that produces a false statement, then the open sentence is not an identity.

But suppose that we have not been able to find a single false substitution.* What do we know then? We *do not* know that the open sentence is an identity. To know this we should have had to try *every* possible substitution, which is impossible, because there are far too many numbers. No one could ever try them all. If we have tried enough numbers to satisfy ourselves, we may believe that the open sentence is probably an identity, but we do not know for sure.

Thus we can sort all the open sentences into two categories: those that we *know are not identities* because we have found a substitution that produces a false statement, and those that we *suspect are identities* since a great many attempts have failed to turn up one single false substitution.

Consider two examples:

(a) How about the open sentence

$$(\square \times \square) + 6 = (5 \times \square)?$$

Is it an identity? That is to say, will it always be *true* for every correctly made substitution?

It is *not* an identity. This is proved by substituting 0 into every box, which results in the false statement $0 + 6 = 0$; hence, $(\square \times \square) + 6 = (5 \times \square)$ is *not* an identity.

(b) How about the open sentence

$$(\square \times \square) + 2 = (3 \times \square)?$$

Is it an identity?

A true statement results if 1 is substituted into every box.

$$\begin{aligned} (\boxed{1} \times \boxed{1}) &+ 2 = (3 \times \boxed{1}) \\ 1 \quad &+ 2 = \quad 3. \end{aligned}$$

---

* *False substitution* means a substitution, correctly performed according to the rule for substituting, which yields a *false* statement.

But does this prove that $(\Box \times \Box) + 2 = (3 \times \Box)$ *is* an identity? Not at all; to be an identity, it must become true for *every* substitution. A true statement also results if 2 is substituted into every box:

$$\begin{array}{c}(\boxed{2} \times \boxed{2}) + 2 = (3 \times \boxed{2}) \\ 4 \qquad + 2 = \qquad 6.\end{array}$$

Perhaps it really *is* an identity. If we are satisfied, we might add it to our list of tentative identities. But if we do so, we have been satisfied too easily. This open sentence is *not* an identity because a false statement results if 3 is substituted in every box:

$$\begin{array}{c}(\boxed{3} \times \boxed{3}) + 2 = (3 \times \boxed{3}) \\ 9 \qquad + 2 = \qquad 9.\end{array}$$

(c) How about the open sentence

$$\Box \times (\Box + 1) = (\Box \times \Box) + \Box ?$$

Is it an identity?

If 0, 1, 2, and 3 are substituted, the results will be true:

(a) Substituting 0:
$$\begin{array}{ccc}0 \times (0 + 1) & = & (0 \times 0) + 0 \\ 0 \times \quad 1 & = & 0 \quad + 0 \\ 0 & = & 0\end{array}$$

(b) Substituting 1:
$$\begin{array}{ccc}1 \times (1 + 1) & = & (1 \times 1) + 1 \\ 1 \times \quad 2 & = & 1 \quad + 1 \\ 2 & = & 2\end{array}$$

(c) Substituting 2:
$$\begin{array}{ccc}2 \times (2 + 1) & = & (2 \times 2) + 2 \\ 2 \times \quad 3 & = & 4 \quad + 2 \\ 6 & = & 6\end{array}$$

(d) Substituting 3:
$$\begin{array}{ccc}3 \times (3 + 1) & = & (3 \times 3) + 3 \\ 3 \times \quad 4 & = & 9 \quad + 3 \\ 12 & = & 12\end{array}$$

Will it work every time? We cannot be sure. But if you put it on your tentative list of identities, then you will be in agreement with every mathematician you meet. Of course, if you found a single substitution that resulted in a *false* statement, then we would all have to change our minds, for then we would *know* that the open sentence *does not* become true for every possible substitution.

If you have never dealt with identities before, then prior to discussing them much in class, you may want to find some fellow teacher and attempt to convince him that this open sentence

$$(\Box \times \Box) \times [(\Box \times \Box) + 11] \\ = (6 \times \Box) \times [(\Box \times \Box) + 1]$$

is or is *not* an identity. By the time you have finished discussing this, you should understand the point of Chapter 6.

Chapter 6
IDENTITIES AND OPEN SENTENCES

[page 10]

ANSWERS AND COMMENTS

(1) What do we mean by an **identity?**

(1) **An identity is an open sentence that becomes true for every correctly made substitution. (As students frequently put it: "You can substitute *any* number, and the result will be true.")**

Which of the following are identities?

(2) $\square + 3 = 3 + \square$

(2) **This *is* a tentative identity.**

If this *is* an identity, it will be true for *every* substitution.

Let's try a few substitutions (remember, of course, that the rule for substituting must be observed: whatever number is put into one box must be substituted into all the other boxes):

(a) $3 \rightarrow \square$: $\boxed{3} + 3 = 3 + \boxed{3}$
$6 = 6$ True

(b) $0 \rightarrow \square$: $\boxed{0} + 3 = 3 + \boxed{0}$
$3 = 3$ True

(c) $4 \rightarrow \square$: $\boxed{4} + 3 = 3 + \boxed{4}$
$7 = 7$ True

How about fractions?

(d) $\frac{1}{2} \rightarrow \square$: $\boxed{\frac{1}{2}} + 3 = 3 + \boxed{\frac{1}{2}}$
$3\frac{1}{2} = 3\frac{1}{2}$ True

How about negative numbers?

(e) $^{-}8 \rightarrow \square$: $\boxed{^{-}8} + 3 = 3 + \boxed{^{-}8}$
$^{-}5 = {}^{-}5$ True

How about very large numbers?

(f) $1{,}000{,}000 \rightarrow \square$: $\boxed{1{,}000{,}000} + 3 = 3 + \boxed{1{,}000{,}000}$
$1{,}0001003 = 1{,}000{,}003$ True

If you are convinced by now, then put $\square + 3 = 3 + \square$ on your tentative list of identities.

(3) $\square + 4 = 4 + \square$

(3) **This is an identity. (See the discussion for the preceding problem.)**

(4) $\square \times 0 = 0$

(4) **This is an identity.**

(5) $\square \times (\square + 1) = (\square \times \square) + 1$

(5) **This is not an identity.**

If this were an identity, then it would be *true* for every substitution.

Let's try a few:

$$\text{(a)}\quad 1 \rightarrow \square : \quad \boxed{1} \times \left(\boxed{1} + 1\right) = \left(\boxed{1} \times \boxed{1}\right) + 1$$

$$1 \times 2 = 1 + 1$$

$$2 = 2 \qquad \text{True}$$

$$\text{(b)}\quad 2 \rightarrow \square : \quad \boxed{2} \times \left(\boxed{2} + 1\right) = \left(\boxed{2} \times \boxed{2}\right) + 1$$

$$2 \times 3 = 4 + 1$$

$$6 = 5 \qquad \text{False}$$

So now you know! The open sentence

$$\square \times (\square + 1) = (\square \times \square) + 1$$

is not an identity.

This can be an exceptionally satisfying lesson; on some of the problems, the class usually divides into two factions. One faction will claim that it can prove that the open sentence in question is not an identity, and the other faction disputes this. If you enjoy this kind of argument among members of the class, this lesson can be quite exciting.

(6) $3 + \square = 5$

(6) **This is clearly not an identity.**

Any number except 2 can be used to settle the matter.

(7) $2 \times \square = 2$

(7) **This is clearly not an identity.**

(8) $\square \times (\square + 1) = (\square \times \square) + \square$

(8) **This is an identity.**

The proof that this is an identity comes in a later lesson. At this stage, the children should ultimately decide to include this on their tentative list of identities. (Later on they will come to regard this as a special instance of the *distributive law*, but they are not ready for that just yet.)

(9) $\square \times (3 + \square) = (3 \times \square) + \square$

(9) **This is not an identity.**

You should have no difficulty in finding a false substitution (i.e., one that produces a false statement).

(10) $(\square \times \square) + 6 = 5 \times \square$

(10) **This is not an identity.**

This is actually a quadratic equation with the truth set $\{2, 3\}$. Hence, any number other than 2 or 3 will produce a false statement.

(11) Jerry says that

$$(\square \times \square) + 6 = 5 \times \square$$

is an identity because he substituted 2 and the result was a true statement.

What do you think?

(11) **If you substitute 2, you *do* get a true statement. This, however, does *not* prove that**

$$(\square \times \square) + 6 = (5 \times \square)$$

is an identity; to be an identity it must become true for *every* number that we substitute, and we have only seen that it becomes true if we substitute 2.

Questions 11 through 14 have a point. They are intended to show the children the danger of generalizing from instances, and to point out why, with the logic available to the children at this stage, they cannot prove that an open sentence is an identity.*

The method of reasoning known as *induction* or *generalizing from instances* is not allowed as part of the legal logical structure of formal mathematics. Nonetheless, *generalizing from instances* is probably the most fruitful method for making original discoveries in mathematics, even though it is not allowable in any effort to *prove* them.

This is perhaps almost a philosophical point; it is possibly the subtlest matter in this entire book. If you feel unsure about it, talk it over with some university mathematician in your area.

(12) Helen says that just trying one number isn't enough.
What do you think?

(12) **See answer to question 11 preceding.**

(13) Jerry says that if you try 3, the result will be true.
Do you agree?

(13) **If you substitute 3, you *do* get a true statement. This, however, does *not* prove that**

$$(\square \times \square) + 6 = (5 \times \square)$$

is an identity. To be an identity, the open sentence must have a truth set consisting of *all* numbers. Thus far, all we know (compare the answer to question 11) is that the truth set for

$$(\square \times \square) + 6 = (5 \times \square)$$

includes at least the two numbers 2 and 3.

(14) Is $(\square \times \square) + 6 = 5 \times \square$ an identity?

(14) **No**

(15) What do we mean by an identity?

(15) **An *identity* is an open sentence that becomes true *whatever* number is substituted into the boxes, provided the *same* number is substituted into every box.**

(16) Is $3 + \square = 5$ an identity? How many numbers do you have to try?

(16) **No. All you need is to find a single *false* substitution (i.e., a substitution that produces a false statement) in order to prove conclusively that the equation is not an identity.**

(17) How can you prove that

$$\square + 1 = \square$$

is **not** an identity?

(17) **The first number substituted will settle the matter. The truth set for the open sentence**

$$\square + 1 = \square$$

---

* Actually, there is an exception to this: the children *can* prove that any "trivial" open sentence (exactly the same on both sides, such as $\square + 3 = \square + 3$) is an identity and you may be satisfied with their proofs of a few very basic identities (such as $\square \times 0 = 0$).

is the empty set, {▨}. That is, there is no number that will produce a *true* statement.

(18) How can you prove that

$$(\square + 1) \times (\square + 1) = (\square \times \square) + (\square + 1)$$

is an identity?

(18) **This is *not* an identity. It can be proved by finding a substitution that will produce a false statement; any number except 0 will do this.**

This, of course, is a trick question to make sure that the children's evaluation of any open sentence is based on intrinsic evidence contained in the open sentence itself and not upon cues which the teacher or the workbook inadvertently supply. In teaching, this means a "poker" voice must be used in an attempt not to give away the answer by tone of voice, or inflection, or the form of questions. In terms of the student text this means that questions are included, such as the present one, where the children are asked to do something that is impossible. They soon learn to be alert and to *think* about every question.

Chapter 7
THE POINT-SET GAME

[page 11]

To play this game, you say two numbers.

The first number goes in the ☐.

The second number goes in the △.

For example: If you said "2, 3," that would mean,

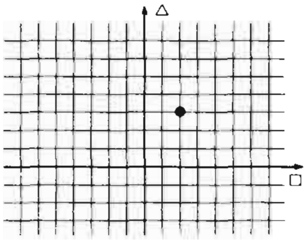
$\boxed{2}, \overset{\triangle}{3}$.

Now we mark a point on a graph to represent the numbers.

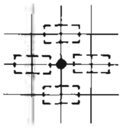
Points for one team are marked ●.

Points for the other team are marked ○.

Example: If the ● team said "2, 3," it would be marked:

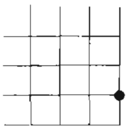


A point is connected to "**breathing spaces**."



This point is connected to **four** breathing spaces.

(1) This point is at the edge of the board.



It is connected to how many breathing spaces? Where are they?

chapter 7 / *Pages 11–13 of Student Discussion Guide*

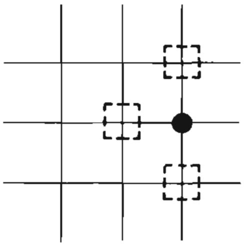
THE POINT-SET GAME*

The point-set game is actually the Japanese game of Go. In order to simplify scoring, the authentic rules are changed in one way only. This modified version is scored by counting the number of points still on the board at the end of the game. The team with the larger number of points wins.

The rules are very simple, with two exceptions. The rule for playing into an "eye" and the rule for replaying a single "dead stone" are rather subtle. We shall try to show all the rules by means of excerpts from games.

ANSWERS AND COMMENTS

(1) This point is connected to *three* breathing spaces, as shown:



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* The point-set game discussed in this chapter is a modified version of the Japanese game of Go. The original Japanese game of Go is extremely subtle and difficult. The modified version is very much simpler, but it is nonetheless a moderately sophisticated game. A still simpler game is desirable for use with younger children—say, in grades 2, 3, or 4. It is sometimes also desirable with older children. Such a game is presented in the film "A Lesson With Second Graders," and is described at greater length in the booklet which accompanies that film. Briefly, this game is a modified version of tic-tac-toe. Points are plotted on the intersections of lines, as is normally done with Cartesian co-ordinates. If the board size permits five points in a straight line, an interesting game can be achieved by ruling that four points in an uninterrupted straight line constitute a win. Various board sizes, and various rules about how many points one must get in a straight line in order to win, can be made; this permits one to vary the subtlety and difficulty of the game quite a bit. In any of its forms, however, the modified tic-tac-toe game is a good deal simpler than the point-set game described here.

(2) This point is in a corner of the board.

It is connected to how many breathing spaces? Where are they?

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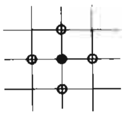
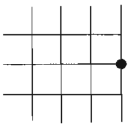
When a point is **surrounded**, it is erased. Example:



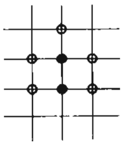
The ● must now be erased.

(3) How many ○ points would you need in order to surround this ● point?

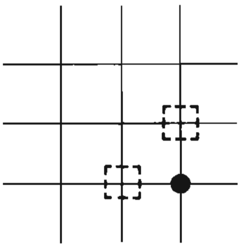


A group of points can be surrounded in just the same way.

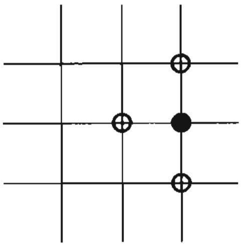
(4) Do the ● points have any breathing space? Where? Must they be erased?



(2) This point is connected to *two* breathing spaces, as shown:

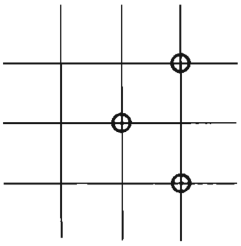


(3) You would need three, placed as shown:

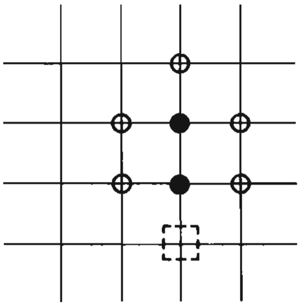


This ● must now be erased,

so the picture becomes:



(4) The ● points do have *one* breathing space left:



Consequently, they would *not* be erased at this stage. (Their prospects for the future, however, are reasonably bleak, especially if it is the ○ points' turn to play next.)

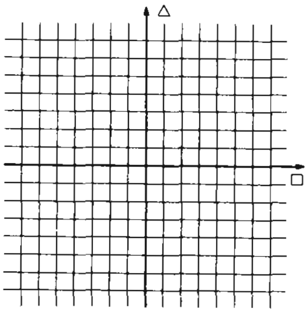
(5) Can you surround ● points by marking one more ○? Where? Must the ● points now be erased?

Remember: We only count connections along the straight lines.

You may use numbers with signs: $^{+}2$, $^{-}3$.

If you do not give the sign (for example: 1, 2), that is the **same** as a **positive** sign ($^{+}1$, $^{+}2$).

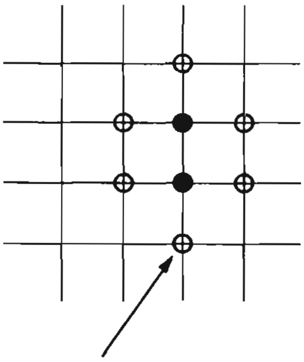
Sample game: Suppose we watch the first few moves of a game.



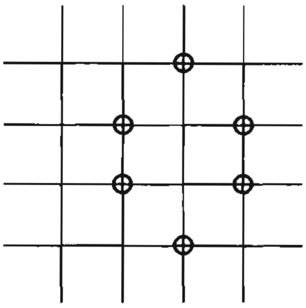
● team: 2, 2

(6) Where would you mark this point?

(5) If it is now the ○'s turn to play, they can place a "○" as shown:



The solid dots are now surrounded (i.e., they have no breathing space left), and the group of ●'s must be erased, so that the picture becomes:

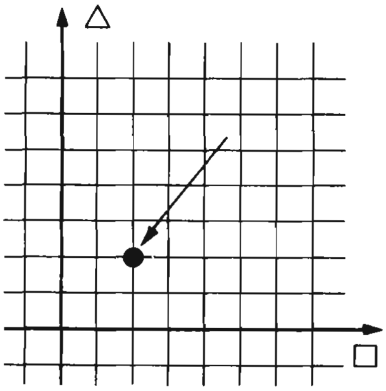


(6) We count exactly as in ordinary co-ordinate geometry. The *first* number goes into the placeholder box, the *second* into the placeholder triangle. (Just as (2, 3) in ordinary analytic geometry means x = 2, y = 3.) We count from the heavy lines (i.e., the "x-axis" and "y-axis").

Consequently, the point 2, 2 would be

$\boxed{2}$, △2

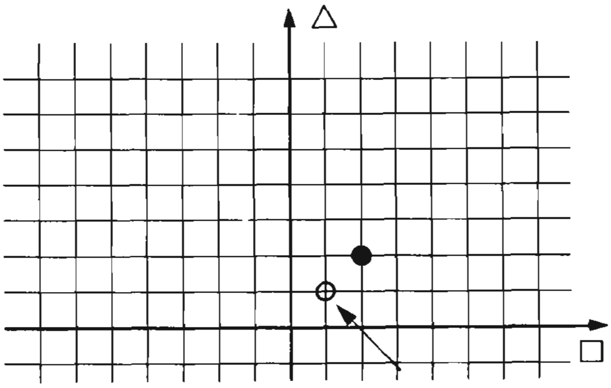
and would be located here:



○ team: 1, 1

(7) Where would you mark this point?

(7)

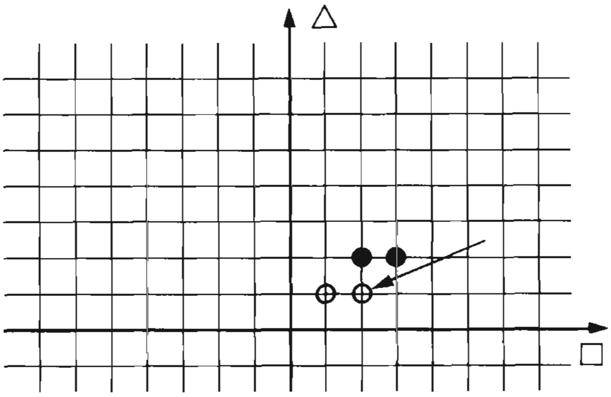


[page 13]

● team: 3, 2

(8) Where would you mark this point?

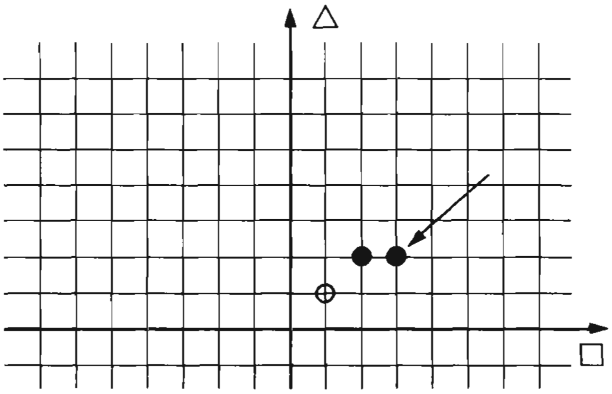
(8)



○ team: 2, 1

(9) Where would you mark this point?

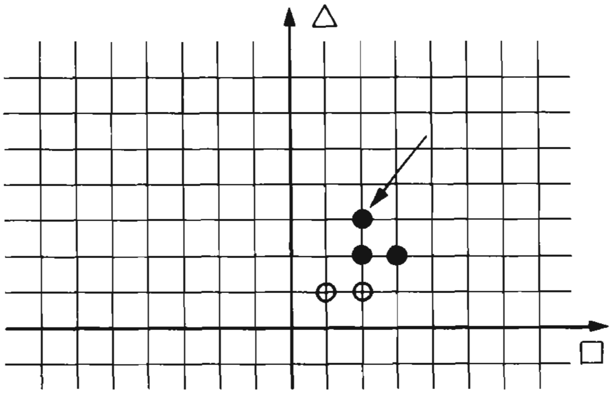
(9)



● team: 2, 3

(10) Where would you mark this point?

(10)

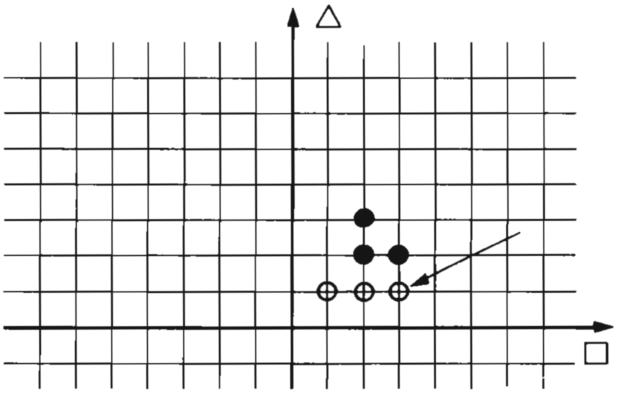


○ team: $^+3$, $^+1$

(11) Where would you mark this point?

Try playing your own game.

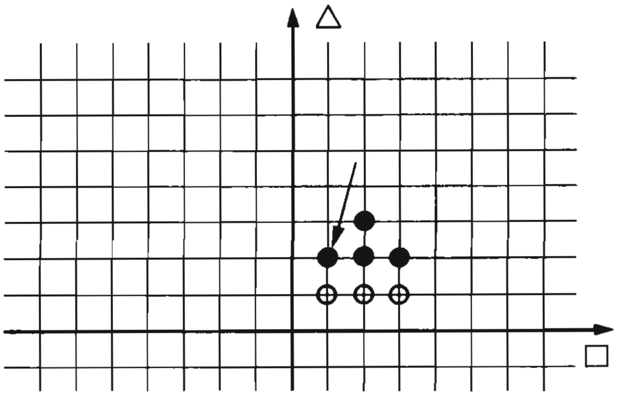
(11)



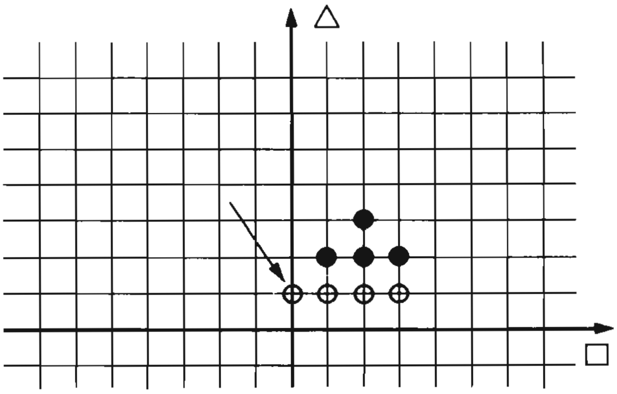
Remember: $^+3$, $^+1$ means the same thing as 3, 1.

Let us continue this game a bit further:

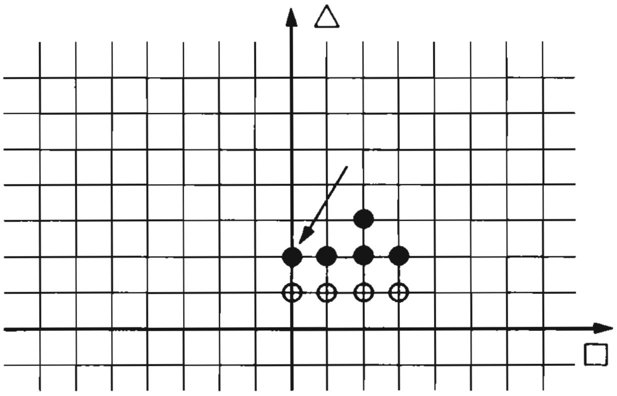
● Team: 1, 2



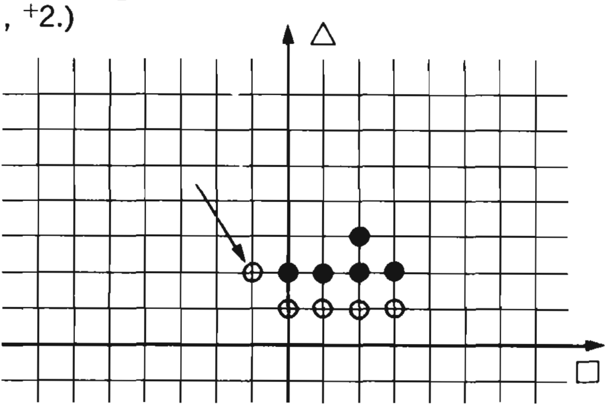
○ Team: 0, 1



● Team: 0, 2



○ Team: $^{-}1, 2$ (Read: negative one, two. This is of course the same thing as $^{-}1$, $^{+}2$.)

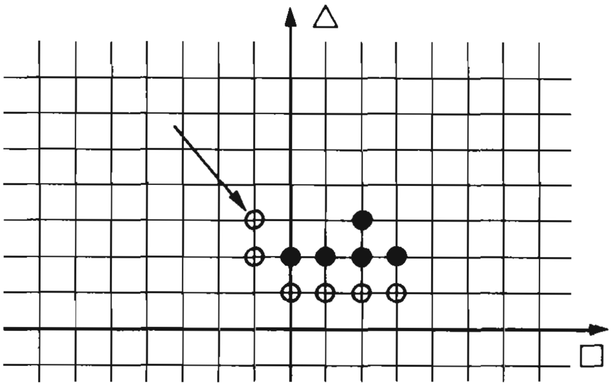


● Team: 4, 2

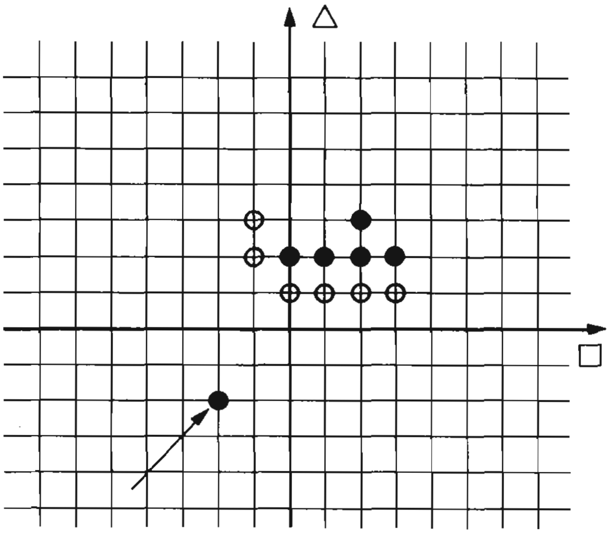
Since this move lies beyond the edge of the board, it is illegal. No point is marked.

Rule on illegal moves: If the intersection lies outside the boundaries drawn (we are presently using $x = \pm 3$, $y = \pm 3$ as our boundaries), the move is illegal. No point is marked, and that player has merely wasted his turn. (You may wish make an exception on the very *first* play of the very first game—i.e., on ● Team's first move.)

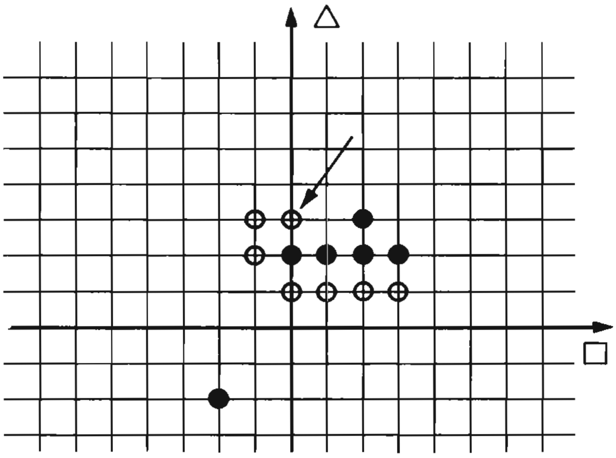
○ Team: $^{-}1, 3$



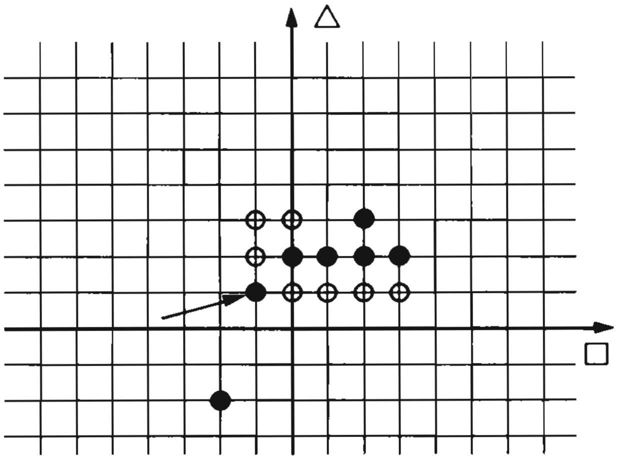
● Team: $^{-}2, ^{-}2$



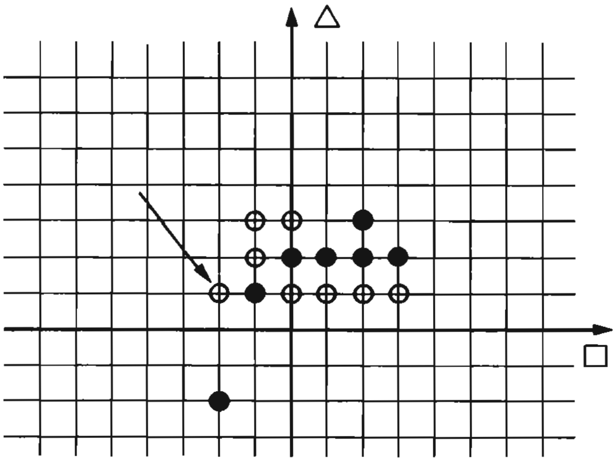
○ Team: 0, 3



● Team: $^-1$, $^+1$



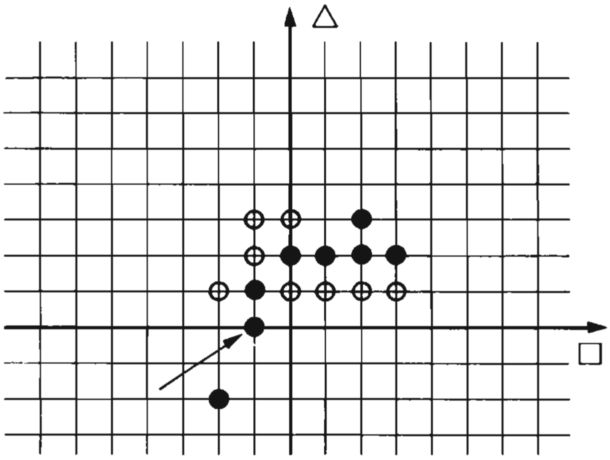
○ Team: $^-2$, 1



Warning rule: Whenever you take away the next-to-last breathing space of an opponent's man (or group of men)—*so that the man or group has only one breathing space left!*—you say *atari!* This is a Japanese word, and plays somewhat the role of announcing *check* in chess. It serves notice on the opponent that his man (or group of men) is in danger of being erased, and must somehow be defended (if it is not already too late!).

At this point, ○ Team announces "atari!" to the ● Team. Their ● point at $^-1$, $^+1$ has only one remaining breathing space, namely at $^-1$, 0.

● Team: $^{-}1$, 0



(● Team has managed to save the threatened ● at $^{-}1$, $^{+}1$.)

○ Team: $^{+}3$, $^{+}3$

This move is illegal.

Rule on playing into eyes: The unoccupied intersection at $^{+}3$, $^{+}3$ has no adjacent breathing spaces; it is completely surrounded by ●'s. It is known as an *eye* belonging to the ●'s. It is illegal for the ○'s to play into such an eye, unless by so doing they erase some of the adjacent ●'s. In the present case, a ○ at $^{+}3$, $^{+}3$ would *not* erase any of the surrounding ●'s, and so this move is illegal.

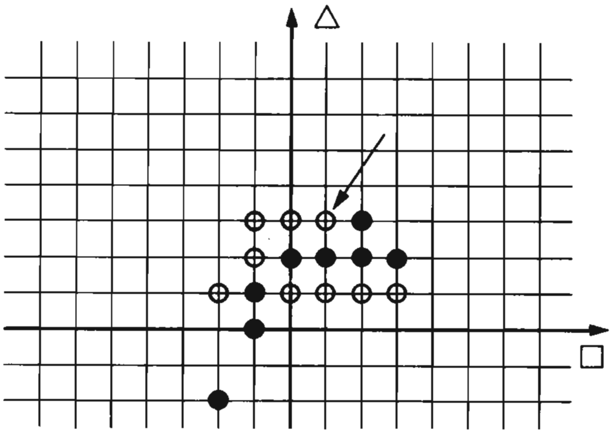
● Team: 2, 1

This move is illegal, since the intersection 2, 1 is already occupied.

Rule on occupied intersections: If an intersection is already occupied (*by either* a ● or a ○), then any further plays at this intersection are illegal. (This would not continue to apply if, at some later time in the game, the intersection became vacant as a result of a point's being surrounded and erased.)

○ Team: 1, 3

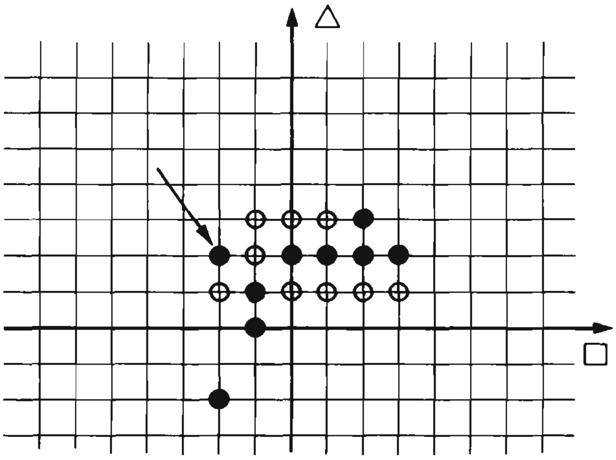
Result:



The ● point group in the upper right-hand corner has only one breathing space left, so ○ Team announces "atari!" to the ● Team. Unfortunately for the ● Team, the warning comes too late. They cannot escape. (Note that ○ Team announced "atari!" at exactly the proper time according to the warning rule.)

● Team: ⁻2, 2

Result:



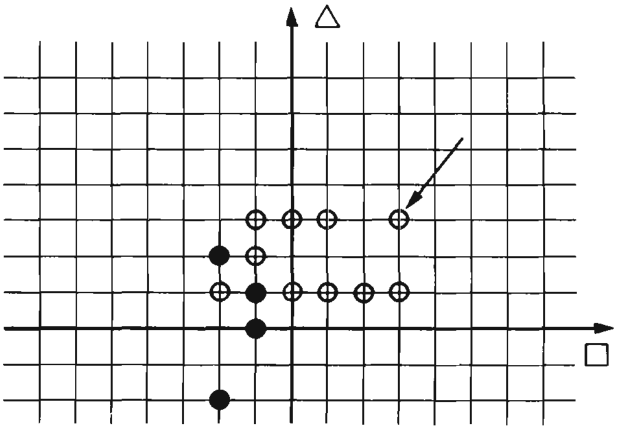
The ○ group (top, center) has only one breathing space left, at ⁻2, 3, so ● Team says "atari!"

○ Team: 3, 3

When this same move was tried a few plays back, it was illegal. Now, however, it is legal.

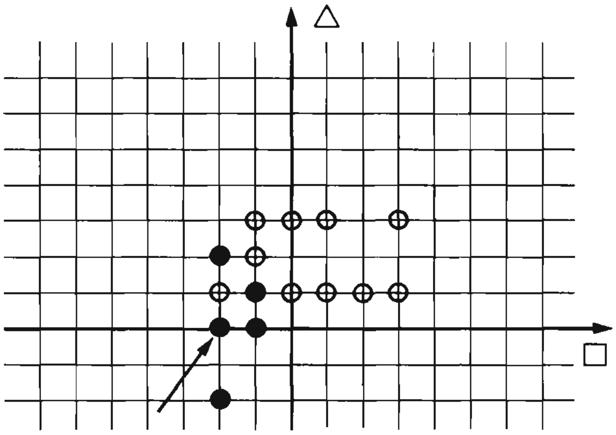
Why? Because *now* when ○ Team puts a point at 3, 3, the ● point group loses the last breathing space, and is erased.

(This is the same rule as in the authentic version of Go. You can study about it, if you wish, in any standard book on Go.)



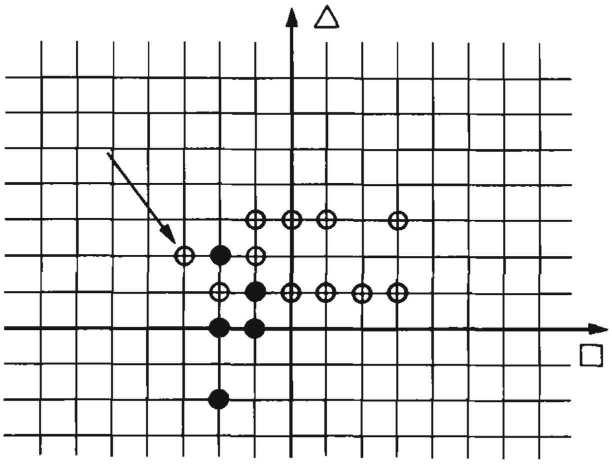
Notice that it was the additional point at 1, 3 that made all the difference between ○ Team's move (⁺3, ⁺3) and ● Team's move.

● Team: ⁻2, 0



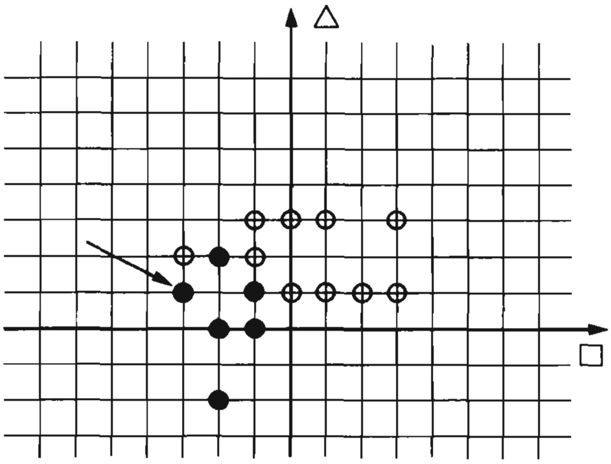
● Team announces "atari!"

○ Team: $^{-}3$, 2



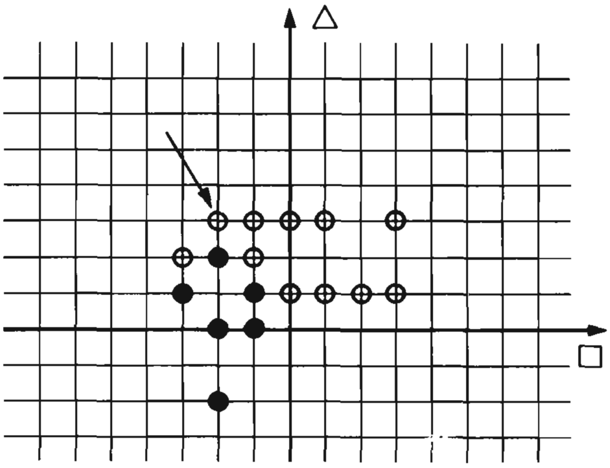
○ Team announces "atari!"

● Team: $^{-}3$, 1



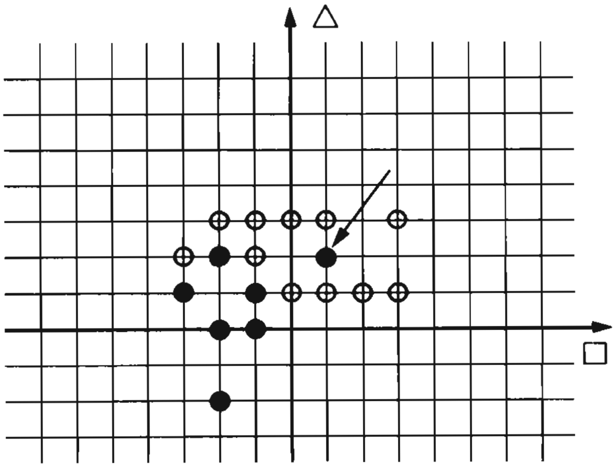
● Team announces "atari!" (Threat to ○ at $^{-}3$, 2.)

○ Team: $^{-}2$, 3



○ Team announces "atari!" (Threat to ● at $^{-}2$, $^{+}2$.)

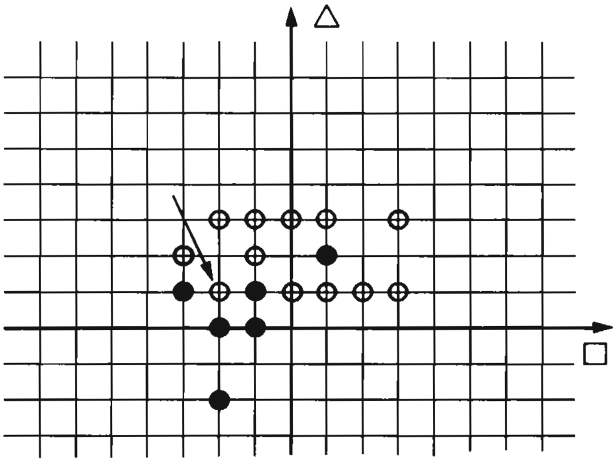
● Team: 1, 2



Observe that ● Team's move *is* legal. Why? There is no rule making it illegal!

○ Team: ⁻2, 1

○ Team is playing into a ● "eye." Nonetheless, this move *is* legal. Why? Because the ● at ⁻2, 2 will be erased!



○ Team announces "atari!" (Threat to ● at ⁻3, 1.)

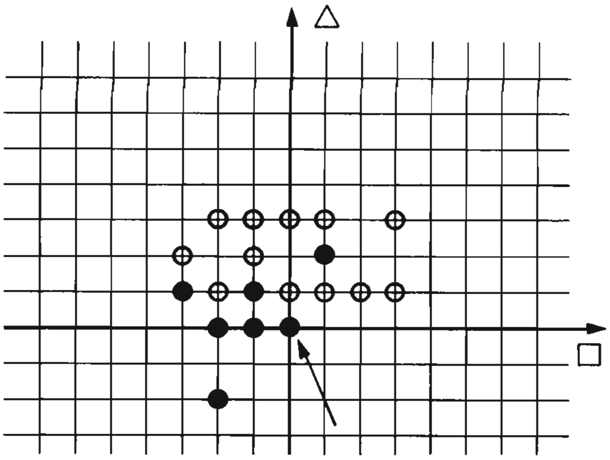
It is now the ● Team's turn to play. Here is the last rule we need to state:

Rule for replaying a single dead stone: If a single ● or ○ has just been erased, it cannot be replayed right back at the same intersection *on the very next play.* It *may* be played back at the same intersection on any *later* play, however, provided that the rule for playing into eyes is not violated thereby.

Of course, the ○ Team may play into its own eye on *its* very next turn, if it wishes to do so. (Of course, the same rule would apply if the roles of ● and ○ were reversed; *all* rules are intended to be symmetric for ○ and ●.)

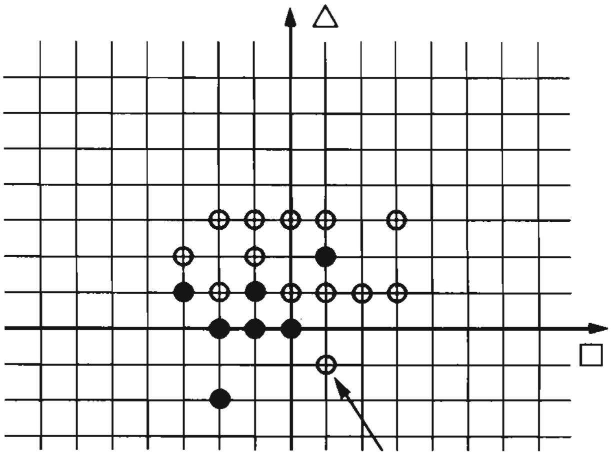
Thus, in the present case, *if* ● Team said ⁻2, 2, this move would be illegal.

● Team: 0, 0



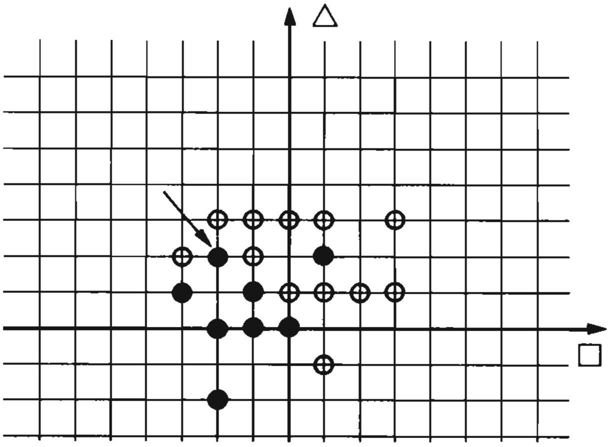
Notice that, at this time, *if* ○ Team said $^{-}2$, 2 that move would be legal.

○ Team: $^{+}1$, $^{-}1$



● Team: $^{-}2$, 2

This move is now legal. It would not have been a legal move for ○ Team, *since it would have been the very next move following the erasure of* a ● at $^{-}2$, 2. For ● Team, one move later on in the game, a play to $^{-}2$, 2 is perfectly legal.



● Team announces "atari!" (because of the threat to the ○ at $^{-}3$, 2).

Notice that ○ Team cannot legally say $^{-}2$, 1, since this would violate the rule on replaying a single dead stone. However, after ○ Team plays elsewhere on the board, it will be legal for ○ Team to play a ○ at $^{-}2$, 1, provided the situation has not changed in such a way as to make this illegal for some other reason.

Nearly every possible situation has now been dealt with.

chapter 8 / *Pages 13–15 of Student Discussion Guide*

OPEN SENTENCES AND SIGNED NUMBERS

A feature of the Madison Project materials is that a topic is not disposed of in one single lesson—perhaps not even in two or three consecutive lessons. This, too, is a matter of the light touch. A topic is returned to repeatedly, and lightly, without requiring *every* child to achieve a certain level of accomplishment by some arbitrary date.

This chapter, for example, returns to the topic of quadratic equations. Every child is not expected to have found the secrets by now. Some people never learn about quadratic equations, and quadratic equations are not among the minimum essentials for productive adulthood—or for promotion into the sixth grade.

We would suggest this attitude toward a student who has not yet found the secret: Quadratic equations are a game. If the students have fun with them (most students do) and if they are good at them (most students are), why, that is very nice. But if the students do not enjoy them and are not good at them, don't worry. There are lots of other things—linear equations, signed numbers, graphs, identities, derivations, and so on—that they will find exciting and amusing.

Pedagogically, it is desired that the students get *experience* with mathematical material, learn from this experience, and enjoy as much of it as possible.

We are convinced that students do much better work when they are kept away from too much pressure. They pull us along after them; we don't push them.

After these chapters were written it was discovered that many classes are able to handle equations with signed-number roots much earlier than originally realized. In several demonstration classes, it was found that the children could solve equations such as those in questions 43 through 48 as early as the first lesson, after working with the pet store problem and a few quadratic equations. Indeed, they could handle harder ones, such as $^{+}10 + \square = {}^{-}2$, and even equations involving products. It is a mistake to expect that we will ever *know* exactly what should be taught in grade 4, or in grade 5, and so on. The teacher who is not surprised by his students' ability is probably not observing them carefully (or sympathetically) enough.

Chapter 8
OPEN SENTENCES AND SIGNED NUMBERS

[page 13]

ANSWERS AND COMMENTS

Can you find the truth set for each open sentence?

(1) $7 + \square = 10$	(1) $\{3\}$
(2) $3 + \square = 60$	(2) $\{57\}$
(3) $3 + (2 \times \square) = 15$	(3) $\{6\}$

(4) $3 + (2 \times \square) = 53$

(4) {**25**}

(5) $7 + (2 \times \square) = 109$

(5) {**51**}

(6) $1 + (3 \times \square) = 34$

(6) {**11**}

(7) $1 + (2 \times \square) = 102$

(7) {$\mathbf{50\frac{1}{2}}$}

You may wish to work on problem 7 by making two lists, one headed *too small* and the other headed *too large*.

Teacher writes on the board:

Student: Try 51.

Teacher: Two times 51 is 102 . . .

$$1 + \underbrace{(2 \times \boxed{51})}_{102} = 102$$

and then we have to add one . . .

$$\begin{array}{l} 1 + (2 \times \boxed{51}) = 102 \\ 1 + \quad 102 \end{array}$$

one plus 102 equals 102—true or false?

$$\begin{array}{l} 1 + (2 \times \boxed{51}) = 102 \\ 1 + \quad 102 \quad = 102 \end{array}$$

Student: False!

Teacher: Is 51 too large or too small?

Student: Too large.

too small	too large
	51

Similarly, the students will find that 50 is too small.

Teacher writes:

too small	too large
50	51

(8) $1 + (2 \times \square) = 62$

(8) {$\mathbf{30\frac{1}{2}}$}

Here, again, you may wish to make two lists, one headed *too large* and the other headed *too small*. It is also very effective to picture this on a number line.

(9) $4 + (2 \times \square) = 7$

(9) {$\mathbf{1\frac{1}{2}}$}

(10) $5 + (2 \times \square) = 10$

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(10) {$\mathbf{2\frac{1}{2}}$}

(11) $5 + (3 \times \square) = 18$

(11) {$\mathbf{4\frac{1}{3}}$}

(12) $8 + (2 \times \square) = 28$

(12) {**10**}

(13) $5 + (3 \times \square) = 20$

(13) {**5**}

(14) $107 + \square = 109$

(14) {**2**}

(15) $103 + (2 \times \square) = 203$

(15) {**50**}

(16) $105 + (2 \times \square) = 207$	(16) **$\{51\}$**
(17) $(\square \times \square) - (5 \times \square) + 6 = 0$	(17) **$\{2, 3\}$**
(18) $(\square \times \square) - (13 \times \square) + 22 = 0$	(18) **$\{11, 2\}$**
(19) $(\square \times \square) - (15 \times \square) + 26 = 0$	(19) **$\{2, 13\}$**
(20) $(\square \times \square) - (8 \times \square) + 15 = 0$	(20) **$\{3, 5\}$**
(21) $(\square \times \square) - (107 \times \square) + 700 = 0$	(21) **$\{7, 100\}$**
(22) $(\square \times \square) - (28 \times \square) + 75 = 0$	(22) **$\{25, 3\}$**
(23) $(\square \times \square) - (15 \times \square) + 50 = 0$	(23) **$\{5, 10\}$**
(24) $(\square \times \square) - (11 \times \square) + 30 = 0$	(24) **$\{5, 6\}$**
(25) $(\square \times \square) - (17 \times \square) + 30 = 0$	(25) **$\{15, 2\}$**
(26) $(\square \times \square) - (3 \times \square) + 2 = 0$	(26) **$\{2, 1\}$**
(27) $(\square \times \square) - (21 \times \square) + 20 = 0$	(27) **$\{1, 20\}$**
Can you make up a quadratic equation for the other people in your class to try to solve? (28) $(\square \times \square) - (\underset{\uparrow}{\_\_} \times \square) + \underset{\uparrow}{\_\_} = 0$	(28) **You may want to suggest to the children that they should *first* decide on the *answers* (i.e., the truth set) that they want, and *then* determine the coefficients. This is a pretty broad hint, but it doesn't give away the secret.**
Can you make up a pet store story for each problem? What answer do you get?	The story answers to problems 29 through 38 will depend on your class. Hypothetical stories are given for problems 29 through 31.
(29) $15 - 10 + 5 = ?$	(29) **A man bought a dog for \$15. Then a woman returned a parakeet and we gave her back \$10.** **(At this point we had \$5 *more* than when we opened up this morning: $15 - 10 = {}^{+}5$.)** **Then a little girl came and bought a kitten for \$5.** **(At this point we had \$10 *more* than when we opened up this morning: $15 - 10 + 5 = {}^{+}10$.)**
(30) $15 - 5 + 1 = ?$	(30) **A man bought a trained rattlesnake named Teddy. He paid us \$15. Then a man brought back a black goldfish, and we gave him back \$5.** **(At this point we had \$10 more in the cash register than we had when we opened up this morning: $15 - 5 = {}^{+}10$.)** **Then somebody came in and bought some canary seed for \$1.** **(At this point we had \$11 more than when we opened up this morning: $15 - 5 + 1 = {}^{+}11$.)**
(31) $15 - 20 + 5 = ?$	(31) **A boy came in and bought a small chimpanzee for \$15. Then a girl came in and brought back a standard-sized poodle, and we gave her back \$20.** **(At this point we had \$5 *less* than when we opened up this morning: $15 - 20 = {}^{-}5$.)** **The response, ${}^{-}5$, is read negative five.**

chapter 9 / *Pages 15–16 of Student Discussion Guide*

NUMBERS WITH SIGNS

Perhaps the main point of this chapter is to get the children thinking in terms of a *number that has a sign as an integral part* (e.g., $^{+}2$) as opposed to the quite different notion of an *unsigned* number (5) which may appear beside an operation sign, such as: 3×5 or $10 + 5$ or $8 - 5$.

Our two main interpretations of *signed numbers* are:

(a) points on a number line, and
(b) gains and losses.

The point of view of the number line (which may be either horizontal, with positive numbers on the right, or else vertical, with positive numbers up and negative numbers down) is presented in Ruth's answer to question 2 below.

The point of view of positive numbers used to represent gains and losses (increases or decreases, hotter or colder on a thermometer, etc.) was presented in the preceding chapter, especially in connection with pet store stories.

Both points of view will reappear from time to time in future lessons.

Chapter 9
NUMBERS WITH SIGNS

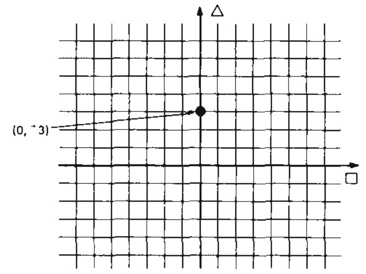
[page 15]

(1) Mathematicians sometimes use numbers with signs, such as:

$^{+}3$	(positive three)
$^{-}2$	(negative two)
$^{+}1965$	(positive 1965)
$^{-}3\frac{1}{2}$	(negative $3\frac{1}{2}$)

Can you think of any places where such numbers might be useful?

(2) Ruth says that numbers with signs could be used on a graph, to distinguish **up** from **down.** For example, $(0, {}^{+}3)$ would be:



ANSWERS AND COMMENTS

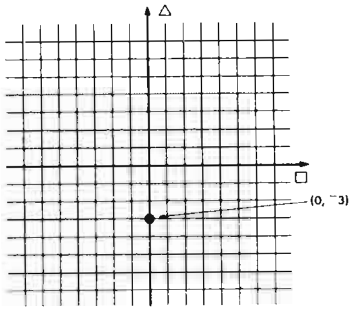
(1) **Graphs; gains and losses; increases and decreases; score for positive and negative teams in the matrix game; temperature on a thermometer; altitude (above and below sea level); charging and discharging a battery (or direction of current flow); time, before and after some crucial time (for example, A.D. and B.C. could be replaced by a suitable use of signed numbers; also, it could be argued that the famous missile-launching countdown, "ten, nine, eight, seven, . . . ," really means: $^{-}10$, $^{-}9$, $^{-}8$, $^{-}7$, $^{-}6$, $^{-}5$, . . .).**

(2) **Yes, Ruth is exactly right!**

You may feel that the rest of this chapter spells things out too explicitly—or is simply unnecessary for your class (or would be unprofitable). In that case, omit it.

whereas, $(0, {}^{-}3)$ would be:

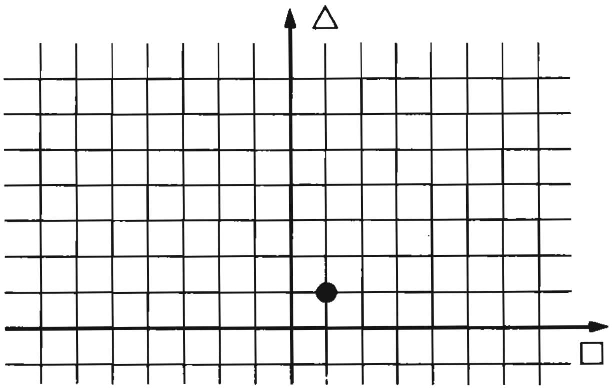
[page 16]



Do you agree with Ruth?

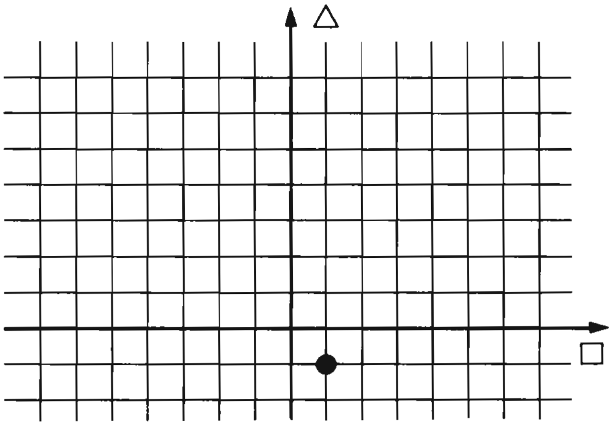
(3) Where would you graph $({}^{+}1, {}^{+}1)$?

(3)



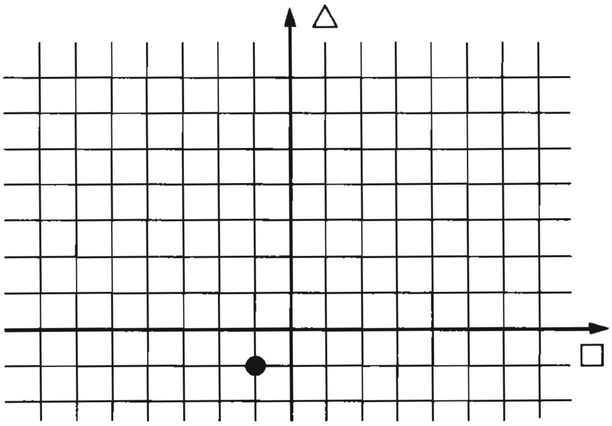
(4) Where would you graph $({}^{+}1, {}^{-}1)$?

(4)



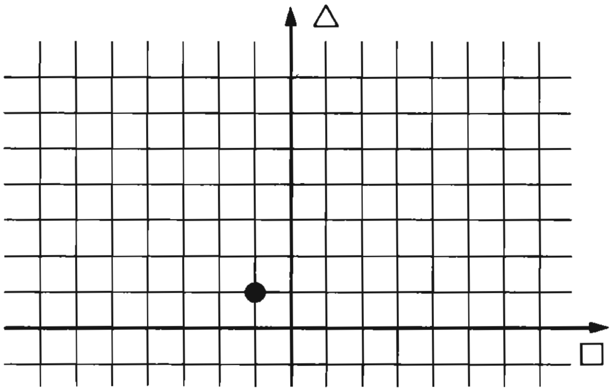
(5) Where would you graph $({}^{-}1, {}^{-}1)$?

(5)



(6) Where would you graph $({}^{-}1, {}^{+}1)$?

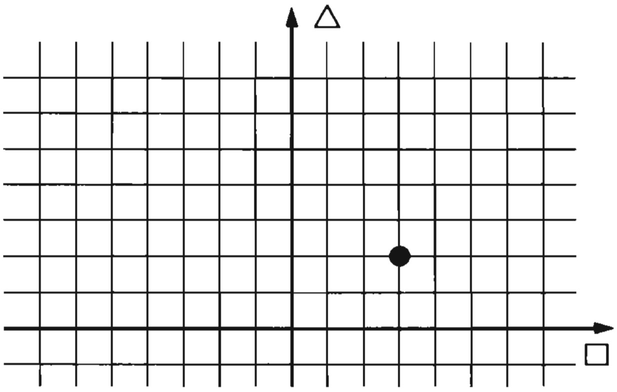
(6)



Where would you graph each of the following points?

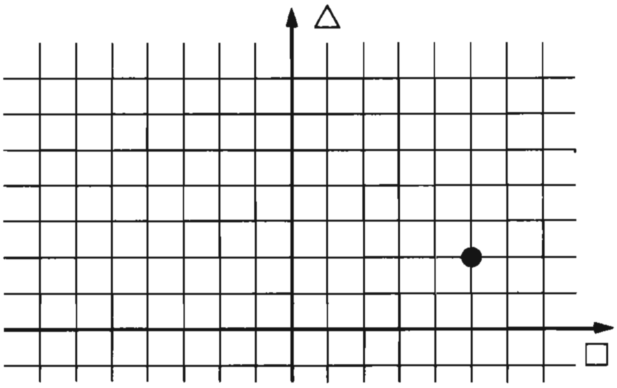
(7) $(^{+}3, ^{+}2)$

(7)



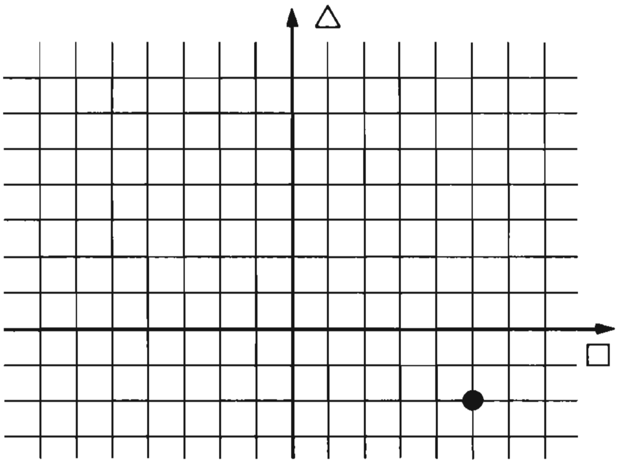
(8) $(^{+}5, ^{+}2)$

(8)



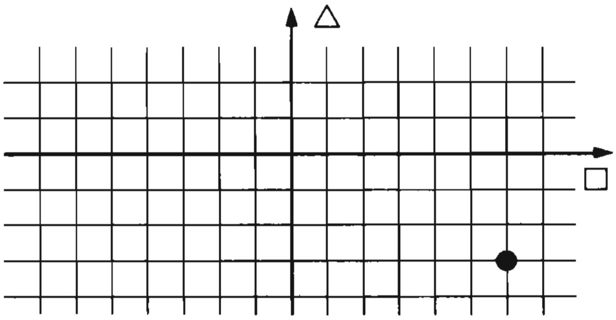
(9) $(^{+}5, ^{-}2)$

(9)



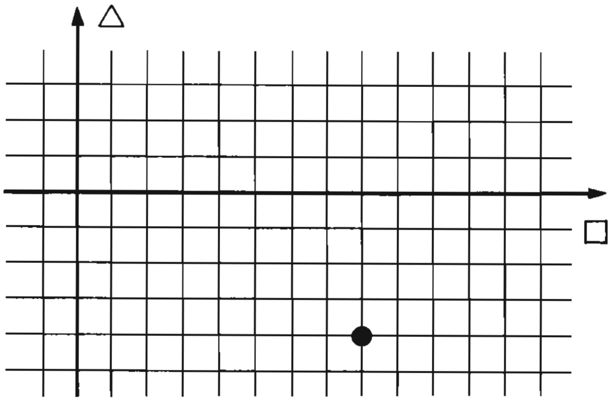
(10) $(^{+}6, ^{-}3)$

(10)



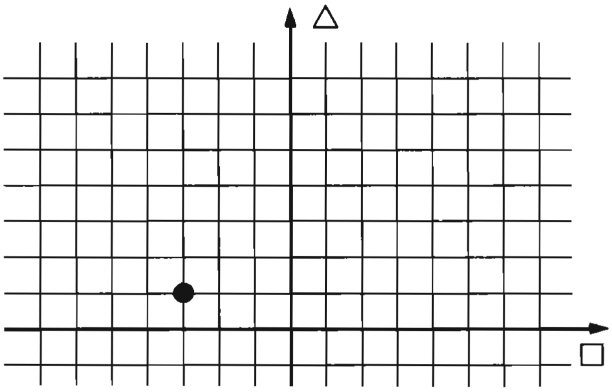
(11) $(^{+}8, {}^{-}4)$

(11)



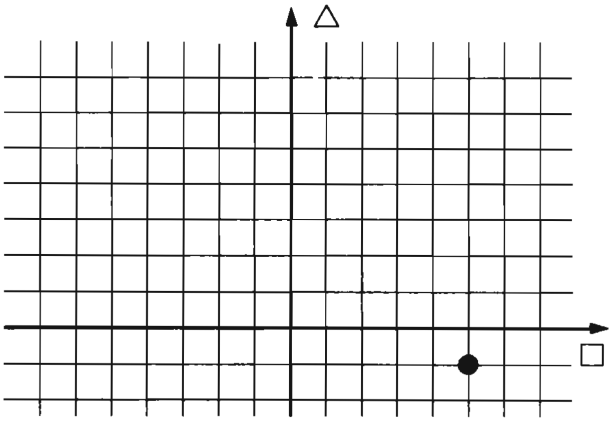
(12) $(^{-}3, {}^{+}1)$

(12)



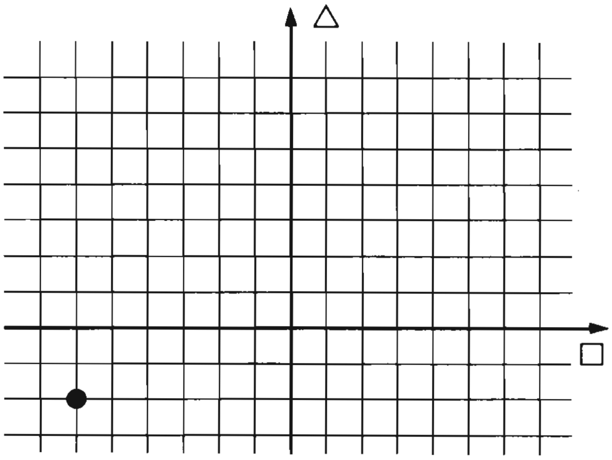
(13) $(^{+}5, {}^{-}1)$

(13)



(14) $(^{-}6, {}^{-}2)$

(14)



chapter 10 / *Page 17 of Student Discussion Guide*

OPEN SENTENCES WITH TWO PLACEHOLDERS*

The point of this chapter (and of the one following) is, of course, the traditional old notion of Cartesian co-ordinates used to display the graph of a function.

The modern language is much clearer than the traditional. If an open sentence has two (different) placeholders, its truth set will consist of *ordered pairs*.

Consider the open sentence $\triangle = \square + 1$. Since a triangle and a box are *different* shapes, we need not put the same number into both, although we *may* do so if we wish.

Each substitution must consist of a *pair* of numbers: one number to go into the box,

$$3 \rightarrow \square: \ \triangle = \boxed{3} + 1$$

and one number to go into the triangle

$$7 \rightarrow \triangle: \ \overset{7}{\triangle} = \boxed{3} + 1.$$

Substituting the pair (3, 7) into the box and triangle, respectively, produces the *false* statement $7 = 3 + 1$, so we know that the pair (3, 7) *does not* belong to the truth set for the open sentence

$$\triangle = \square + 1.$$

Suppose we try the pair (5, 6). The traditional order is *first* to write the number that goes into the box, and *second* the number that is to go into the triangle. By writing the pair as (5, 6), we therefore mean:

$$5 \rightarrow \square$$
$$6 \rightarrow \triangle.$$

Substituting (5, 6) gives the *true* statement $\overset{6}{\triangle} = \boxed{5} + 1$, so the pair (5, 6) *does* belong to the truth set of the open sentence

$$\triangle = \square + 1.$$

Notice that the *order* is important. The pair (5, 6) would mean (according to the order convention stated above):

$$5 \rightarrow \square$$
$$6 \rightarrow \triangle.$$

Hence it produces the *true* statement $\overset{6}{\triangle} = \boxed{5} + 1$; consequently (5, 6) *does* belong to the truth set for $\triangle = \square + 1$.

---

* The idea for this chapter came from Miss Cynthia Parsons, who has been an important member of and contributor to the Madison Project since 1958.

However, the pair (6, 5) would mean

$$\begin{aligned} 6 &\to \square \\ 5 &\to \triangle, \end{aligned}$$

and would produce the *false* statement $\triangle\!\!\!\!\!\!\!{\scriptstyle 5}\;\; = \boxed{6} + 1$. Consequently, the pair (6, 5) does *not* belong to the truth set for

$$\triangle = \square + 1.$$

[Remember that in *sets* the order of listing the elements is *not* important; the set {10, 31} is precisely the same *set* as the set {31, 10}. In *ordered* pairs, however, the order *is* important. The ordered pair (6, 5) is *not* the same as the ordered pair (5, 6).]

How can we list the truth set for an open sentence with two placeholders?

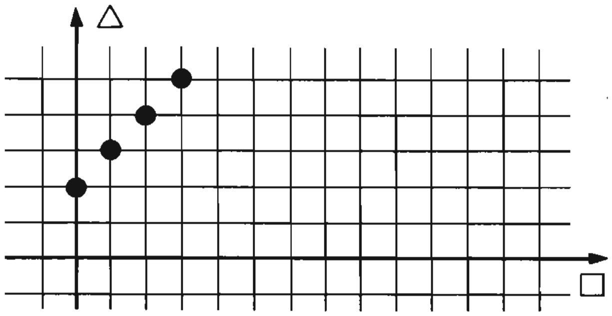
Well, a *set* of numbers can be written as {1, 3, 5}, and a set of *letters* might be written as {A, B, C, W, X, Y}; a *set of ordered pairs* would be written as {(1, 0), (2, 1), (7, 3), (8, 21)}. This is a set with four elements. Each element is an ordered pair. The elements are: (1, 0), (2, 1), (7, 3), (8, 21). An *infinite* set is indicated by a final three dots: {1, 2, 3, 4, . . .} or {(1, 0), (2, 1), (3, 2), (4, 3), . . .}. Using this notation, we could write the *truth set* for the *open sentence* $\triangle = \square + 2$ in the form

$$\{(0, 2), (1, 3), (2, 4), (\tfrac{1}{2}, 2\tfrac{1}{2}), (0.01, 2.01), \ldots\}.$$

Another way to write this same truth set is in the form of a table:

□	△
0	2
1	3
2	4
$\frac{1}{2}$	$2\frac{1}{2}$
7	9
⋮	⋮

In the following chapter, a third method is used for writing a truth set—namely, by means of a graph:



Chapter 10

OPEN SENTENCES WITH TWO PLACEHOLDERS

[page 17]

(1) Can you find a pair of numbers that will make this true?

$$\triangle = \square + 3$$

$$\square, \triangle$$

Ruth says that we can use a **table** to represent the truth set for this open sentence.

$$\triangle = \square + 3$$

(2) Can you fill in the blanks in Ruth's table?

$\square$	$\triangle$
1	4
2	
3	
4	
6	

Table for the Truth Set

(3) Can you fill in this table?

$\triangle = \square + 1$

Open Sentence

$\square$	$\triangle$
0	
1	
2	
3	
$^{-}1$	
$\frac{1}{2}$	
	10
	11

Table for Truth Set

Can you make a table for each truth set?

(4) $\triangle = 2 \times \square$

ANSWERS AND COMMENTS

(1) Here are a few pairs that will produce *true* statements: (1, 4), (2, 5), (3, 6), (0, 3), ($^{-}$1, $^{+}$2) . . .

(2)

$\square$	$\triangle$
1	**4**
2	**5**
3	**6**
4	**7**
6	**9**

(3)

$\square$	$\triangle$
0	**1**
1	**2**
2	**3**
3	**4**
$^{-}$**1**	**0**
$\frac{1}{2}$	$1\frac{1}{2}$
9	**10**
10	**11**

(4) There are very many possible answers. Here is one:

$\square$	$\triangle$
0	**0**
1	**2**
2	**4**
3	**6**
4	**8**
⋮	⋮

(5) $\triangle = (2 \times \square) + 1$

(5) Again, there are many right answers. Here is one:

□	△
0	1
1	3
2	5
3	7
4	9
6	13
⋮	⋮

(6) $\triangle \times \square = 36$

(6) Here is one possible answer:

□	△
1	36
2	18
3	12
4	9
6	6
8	$4\frac{1}{2}$
9	4
10	3.6
12	3
18	2
30	1.2
36	1
72	$\frac{1}{2}$
100	0.36
0.36	100
$\frac{1}{2}$	72
$^{-}1$	$^{-}36$
$^{-}2$	$^{-}18$
$^{-}6$	$^{-}6$
⋮	⋮

← **Remember: it *is* permissible to put the same number into both the box and triangle.**

chapter 11 / *Pages 18–21 of Student Discussion Guide*

GRAPHS*

This chapter continues the discussion of Chapter 10 and provides a third method for writing the truth set of an open sentence with two (different) placeholders: namely, by using graphs.

During the next few lessons, the children should discover the concepts of linear graph, slope, slope coefficient, and the slope-intercept form for writing linear equations.

The subject is not developed in this order, however. Instead, it is approached by taking a "big" and somewhat complicated concept like slope and trying to arrive at it through a sequence of relatively easy discoveries. This can be thought of as breaking down a *compound* concept into its constituent *elementary* concepts.

During this lesson, and the next few lessons following it, it is hoped that the children will discover the following sequence of elementary ideas:

(1) The truth set of an equation involving two placeholders (the box and triangle) can be written in the form of a graph.

(2) There is a *pattern* to this graph.

(3) The pattern is not always exactly the same.

(4) Sometimes the pattern is "over one to the right, up one," or "over one to the right, up two," or "over one to the right, up three," or "over one to the right and up one half," or "over one to the right and down one," etc.

(5) You can tell *which* of these patterns you will get by looking at the equation (the idea of the slope coefficient).

(6) The *pattern* for a truth set is the same, *no matter what size square is used in counting.*

(This idea is not essential at this point. You may prefer to leave it out. It is included only because a sixth grader discovered it by himself, thereby immensely pleasing himself and his teacher. Mathematically this takes you from discrete slope patterns to the slope of a continuous line.)

(7) In an equation like

$$\triangle = ({}^{+}3 \times \square) + {}^{-}2,$$

both the ${}^{+}3$ and the ${}^{-}2$ have a geometrical meaning.

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These ideas can be rephrased in terms of a slightly different (and possibly more useful) sequence of elementary ideas:

Elementary Idea 1—Discrete slope, where the slope is 1.

O: 217 3337909

H: 328 1832

A tape recording or sound film of this activity is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

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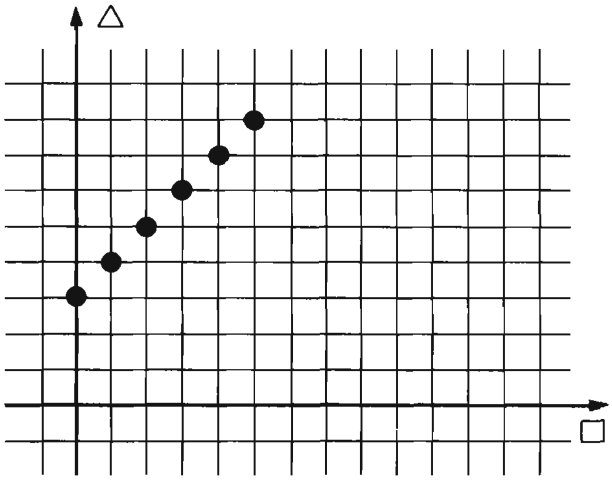
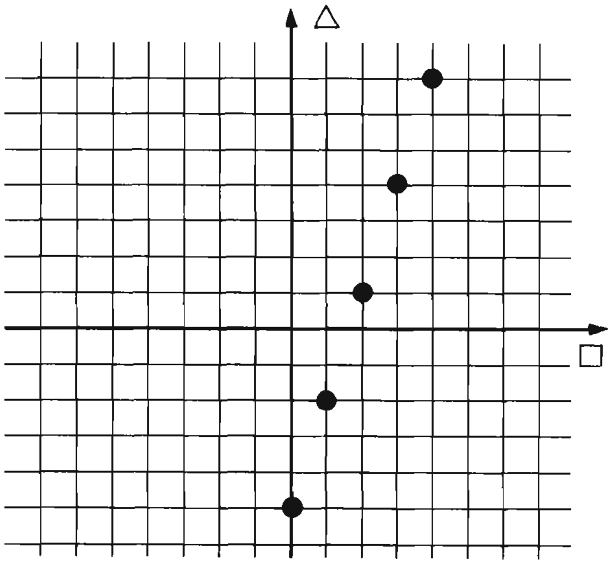
For the equation $\triangle = \square + 3$, a pattern like this is obtained for the truth set:



The pattern is: over one square to the right and up one square.

Elementary Idea 2—Discrete slope, where the slope is a positive integer (in this case actually $^{+}3$).

For the equation $\triangle = ({}^{+}3 \times \square) + {}^{-}5$, a pattern like this is obtained if the truth set is graphed using only whole numbers:



The pattern is: over one square to the right and up three squares.

Elementary Idea 3—Recognition of the slope coefficient.

Actually, the pattern can be determined *even before any points are plotted!*

For the equation

$$\triangle = ({}^{+}4 \times \square) + {}^{-}7,$$

the pattern will be: over one square to the right and up four.

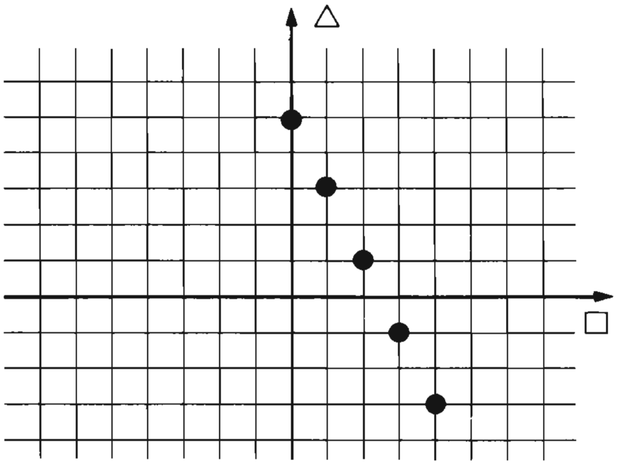
For the equation

$$\triangle = ({}^{+}9 \times \square) + {}^{-}15,$$

the pattern will be: over one square to the right and up nine.

Elementary Idea 4—Discrete slope, where the slope is a negative integer.

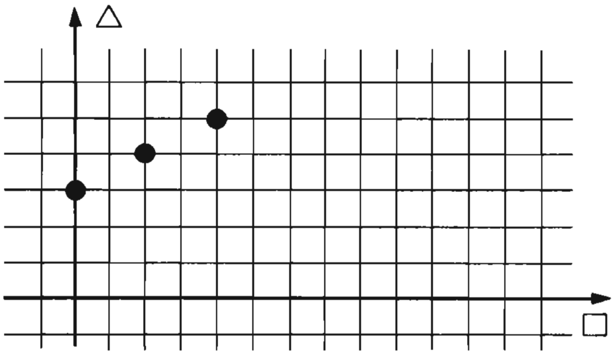
For the equation $\triangle = ({}^{-}2 \times \square) + {}^{+}5$, a pattern like this is obtained for the truth set (using only whole numbers):



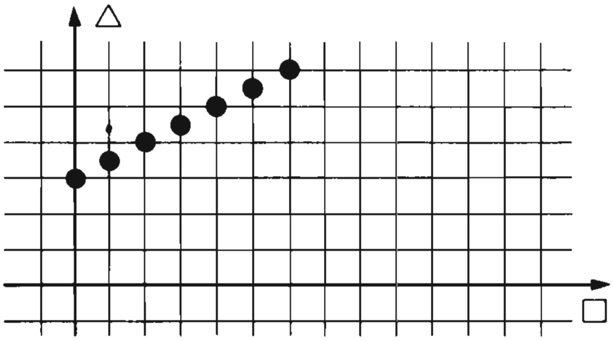
Here, the pattern is: over one square to the right and *down* two.

Elementary Idea 5—Discrete slope, where the slope is a positive fraction.

For the equation $\triangle = (\frac{1}{2} \times \square) + {}^{+}3$, the graph shows this pattern:



The pattern is: over two squares to the right and up one. Or, if more points are filled in,



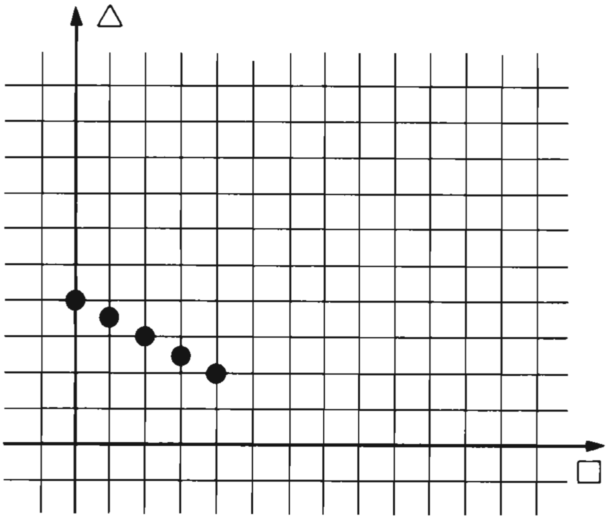
the pattern is: over one square to the right and up one half.

Either way, it is really the same pattern.

(At this stage you may prefer to limit yourself to slopes where the numerator is one, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . ; or, if you prefer, you can tackle the slightly harder case where the slope is $\frac{2}{3}$, $\frac{3}{4}$, and so on.)

Elementary Idea 6—Negative fractional (discrete) slope.

For the equation $\triangle = \left({}^{-}\frac{1}{2} \times \square\right) + {}^{+}4$, the pattern looks like this:



The pattern is: over two squares to the right and down one, or over one square to the right and down one half.

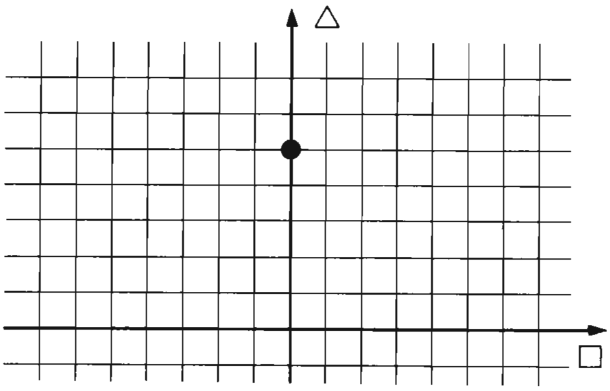
Elementary Idea 7—Where does the line cross the vertical axis?

For the equation $\triangle = \left({}^{+}7 \times \square\right) + {}^{+}5$, if 0 is substituted into $\square$ to get $\triangle = \left({}^{+}7 \times \boxed{0}\right) + {}^{+}5$, the ordered pair (0, $^{+}5$) will produce a *true* statement:

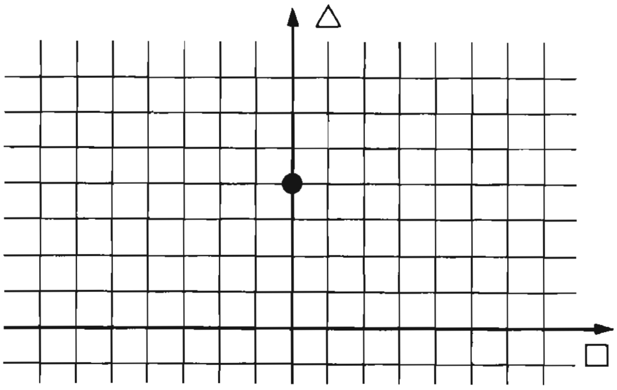
$$\overset{{}^{+}5}{\triangle} = \left({}^{+}7 \times \boxed{0}\right) + {}^{+}5$$

$$5 = 0 + 5$$

On a graph, this looks like this:



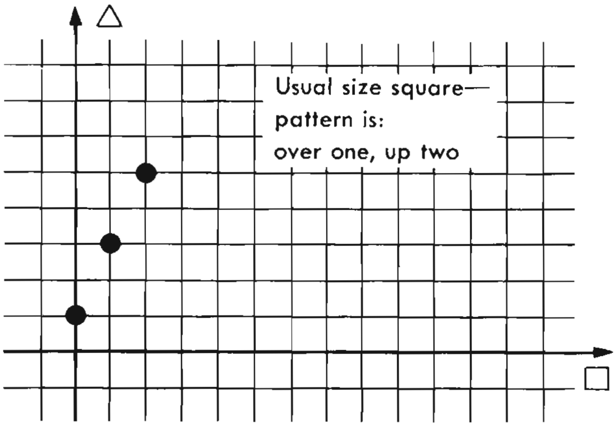
Similarly, for the equation $\triangle = (\frac{3}{2} \times \square) + {}^{+}4$, this point is obtained in the truth set:

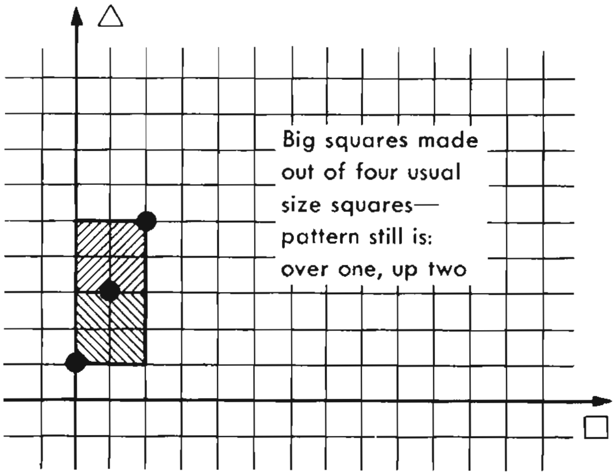


This is a familiar idea from classical analytic geometry.

Elementary Idea 8—Continuous slope.*

If you have a good class and if you yourself are familiar with this material, you might want to introduce Peter's discovery: You can use squares of any size in counting the pattern for the equation $\triangle = ({}^{+}2 \times \square) + {}^{+}1$:





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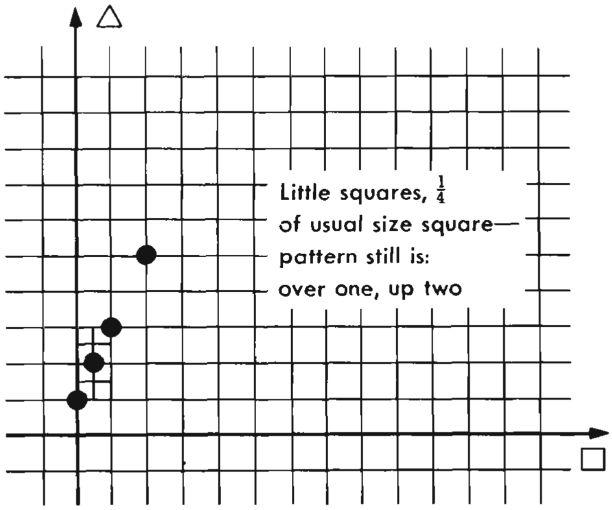
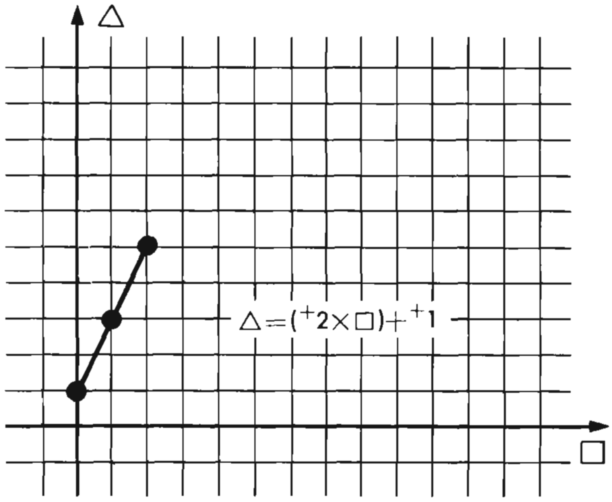
* *Continuous* refers to the solid line, in contrast to *discrete*, which refers to the separate distinct points that were obtained on the earlier graphs as a result of using only whole numbers.



Using squares of any size allows us to fill in all the fractional values, and get the usual continuous line:



Of course, the children cannot pursue this sequence of discoveries to completion in one lesson. It seems advisable to use graphs as part of the day's work for eight or ten lessons (not necessarily consecutive), and by the end of this time most of the children will have discovered most of the elementary ideas that together constitute the compound idea of slope.

You may need to follow your own judgment as to when to use only whole numbers (discrete case) and when to use whole numbers and fractions (continuous case) in substituting into the box.

It is usually easier for the children to discover the slope pattern when working with whole numbers only. It is advisable not to introduce fractions until *after* the children have discovered the slope pattern: over one square to the right and up one square, over one square to the right and up two squares, and so on.

Chapter 11

GRAPHS

[page 18]

(1) Jerry says that he can use a graph to represent the truth set for

$$\triangle = \square + 3.$$

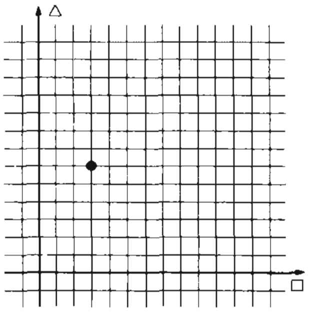
Can you mark four more points on Jerry's graph?

$\triangle = \square + 3$

Open Sentence

$\square$	$\triangle$
3	6
4	7
5	8
6	9

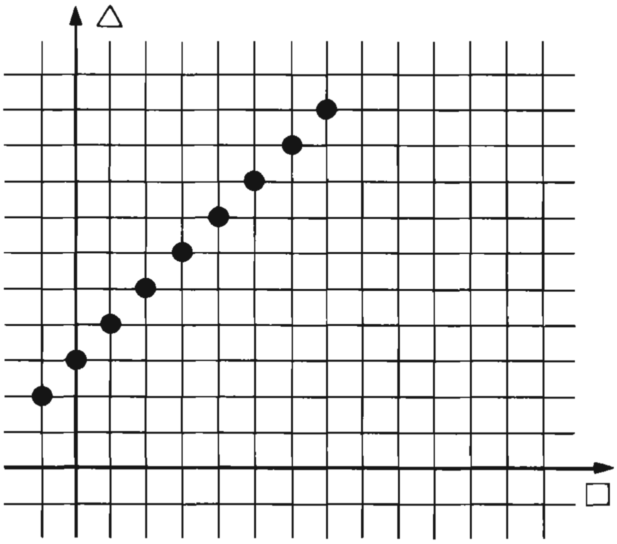
Table for Truth Set



Graph for Truth Set

ANSWERS AND COMMENTS

(1)

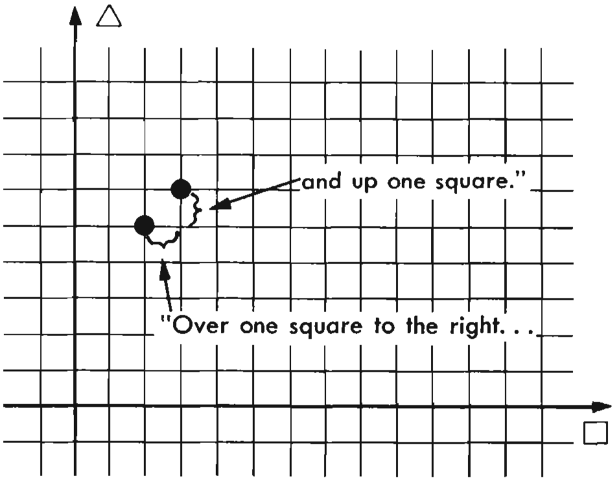


Any of these points would be correct. There are, of course, many others.

It is suggested that you avoid fractions, unless the children themselves wish to use fractions.

It is easier to see the slope pattern if you substitute only whole numbers into the box.

For this equation, of course, the slope pattern is: over one square to the right and up one square.

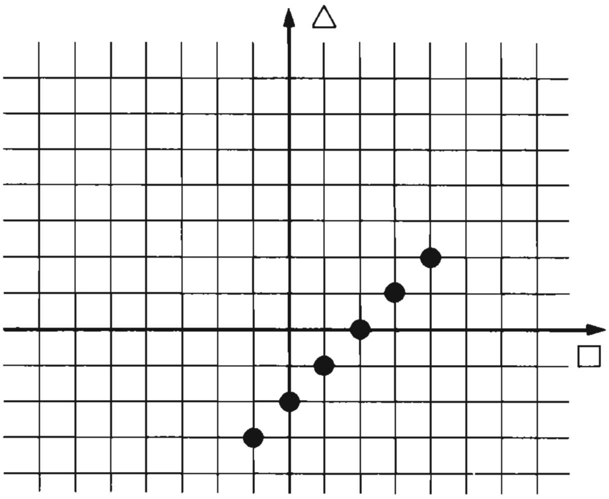


Do *not* tell the children about the slope pattern. You can induce them to discover it by asking them to mark points by looking at the geometry (or the pattern), without doing any arithmetic. At first they may mark points almost at random, but after each point is marked you can find its co-ordinates, substitute into the equation, and see whether the resulting statement is true or false. This helps the children to see the relation between points on the graph and numbers substituted into the triangle and the box. Correct marking of points fulfills the usual definition of *locus:* a point is marked on the graph if and only if its co-ordinates produce a *true statement* when substituted into the open sentence. In past years, perhaps because we gave this notion the Latin name *locus*, it usually seemed very mysterious. Using the currently fashionable language, however, this says only that graphs provide another way of writing the truth set for an open sentence.

(2) Can you make a graph for the truth set of

△ = □ − 2?

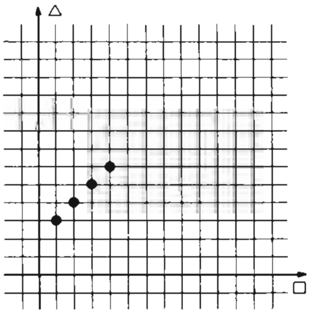
(2)



(There are, of course, many other points besides the six shown.)

(3) Vivian made this graph for the truth set of

△ = □ + 2.



Graph for Truth Set

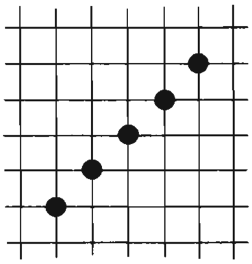
Is Vivian's graph right?

(3) Yes

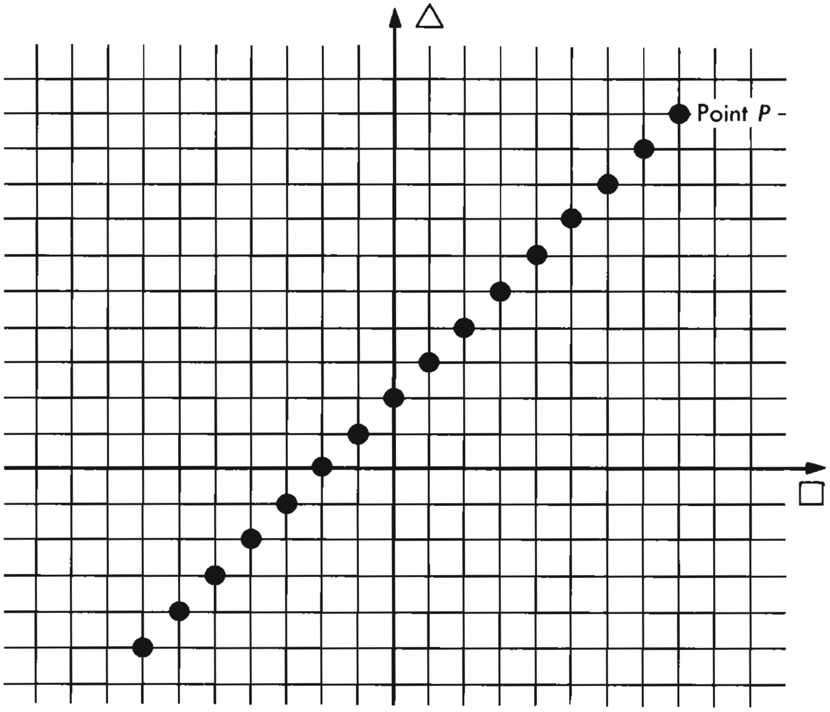
(4) Joe says he can mark another point on Vivian's graph **without** doing any arithmetic. Can you?

(4) You can find more points for Vivian's graph by means of the "ladder" pattern:

Here are some more points for Vivian's graph.

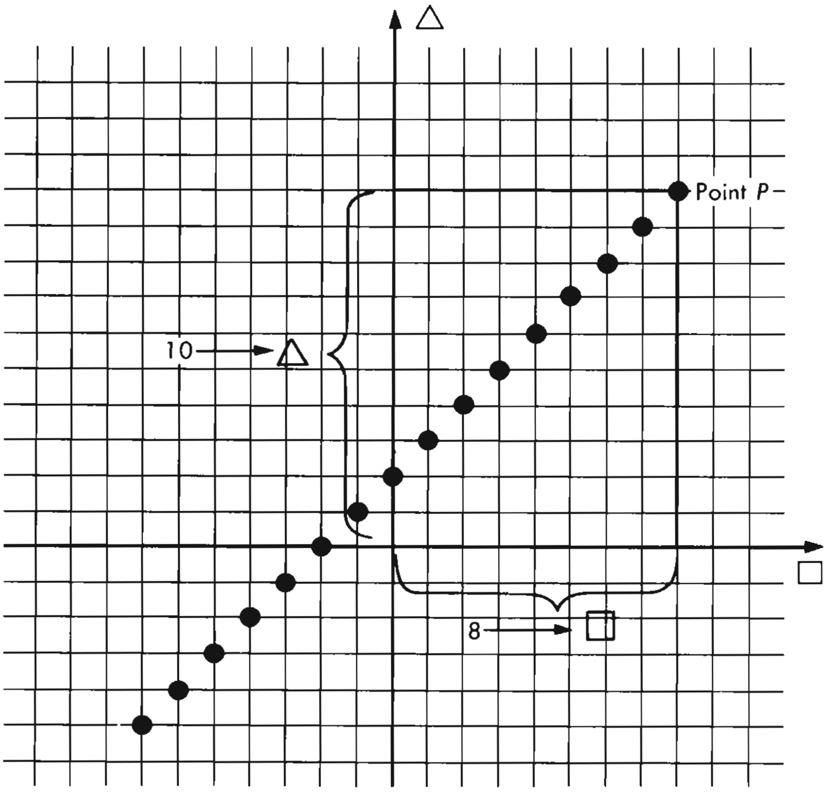


In each case, if you count the co-ordinates of the point, and substitute into the box and the triangle, the resulting statement will be true.



Graph for truth set of $\triangle = \square + 2$

This is illustrated by finding the co-ordinates of the point *P* and substituting into the box and triangle:



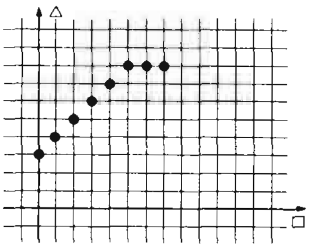
(5) How can you check your point to see if it is really correct?

(6) Can you mark 3 more points on Vivian's graph **without** doing the arithmetic.

[page 19]

(7) Bill made this graph for the truth set of

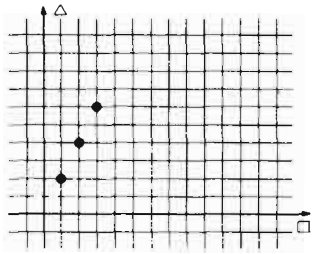
$$\triangle = \square + 5.$$



Is Bill's graph right?

(8) Alice made this graph for the truth set of

$$\triangle = 2 \times \square.$$



Do you agree?

(9) Can you mark 3 more points on Alice's graph **without** doing the arithmetic? (Just look at the geometric pattern.)

If 8 is substituted into the box, and 10 into the triangle, a *true* statement is obtained:

$$8 \rightarrow \square \qquad \underset{10}{\triangle} = \boxed{8} + 2$$
$$10 \rightarrow \triangle$$

(5) You can check a point by substituting, exactly as you did for point *P* in the discussion of question 4.

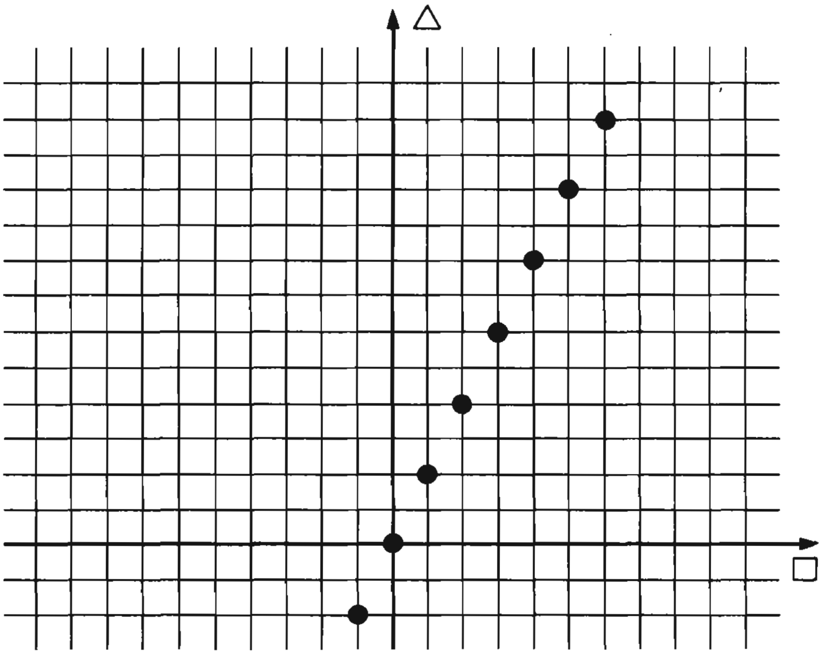
(6) Same as answer to question 4.

(7) No, Bill's graph is not right. You can see this immediately just by looking at his pattern. If you check the points he has marked you will find, in fact, that every one of them is wrong.

(8) Alice's graph is right. (Notice the pattern of over one square to the right and up two squares.)

It seems advisable not to tell this pattern to the children. Indeed, as each child discovers the pattern, encourage him to keep it a secret and let the other children find it for themselves.

(9) Here are some more points. You can find them by using the pattern mentioned in the answer to question 8.

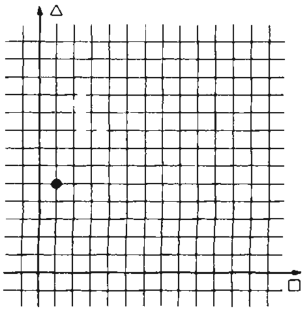


(10) Can you check your points by substituting? Were they right?

(11) Phil started a graph for the truth set of

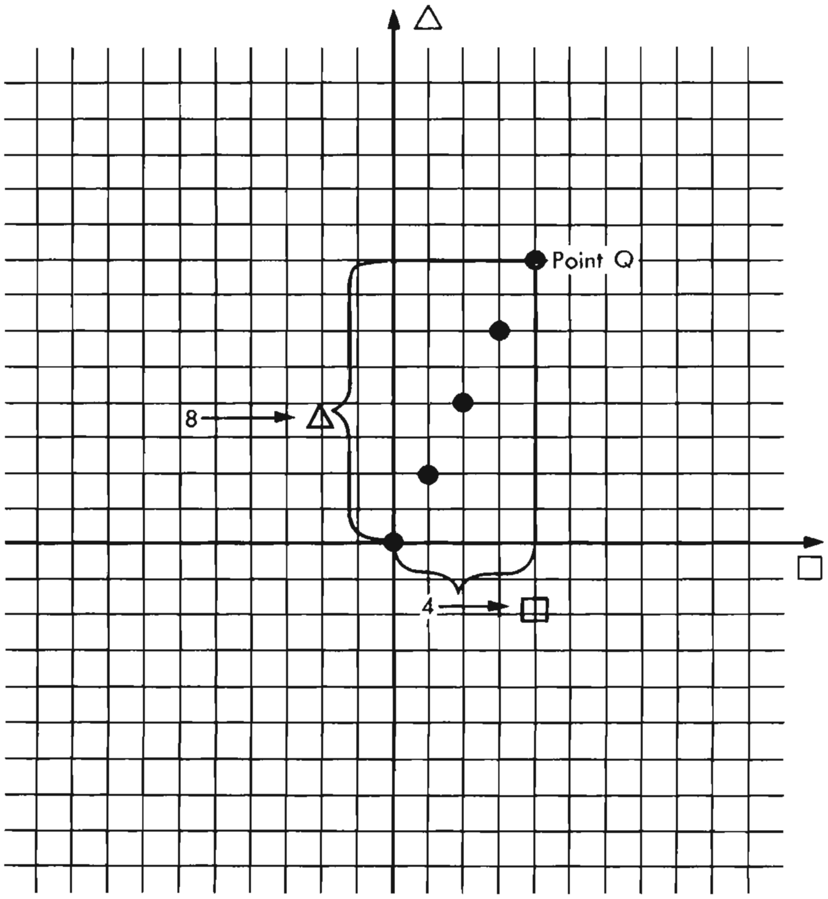
$$\triangle = (3 \times \square) + {}^{+}2$$

this way:



Does Phil's point work?

(10) **Here is a check on *one* point:**



$4 \rightarrow \square$

$8 \rightarrow \triangle$

$$\overset{8}{\triangle} = 2 \times \boxed{4}$$

$$8 = 2 \times 4 \qquad \text{True}$$

The other points could be checked in a similar way; the important *idea* is that this graph is a representation of the *truth set* of the open sentence; if the co-ordinates are substituted into the open sentence, the resulting statement will be *true*. When the co-ordinates (4, 8) of the point *Q* are substituted into the open sentence $\triangle = 2 \times \square$, the result is, indeed, true.

(11) **Yes it does. The co-ordinates of Phil's point are (1, 5); when substituted into Phil's equation, the resulting statement is *true:***

$1 \rightarrow \square$

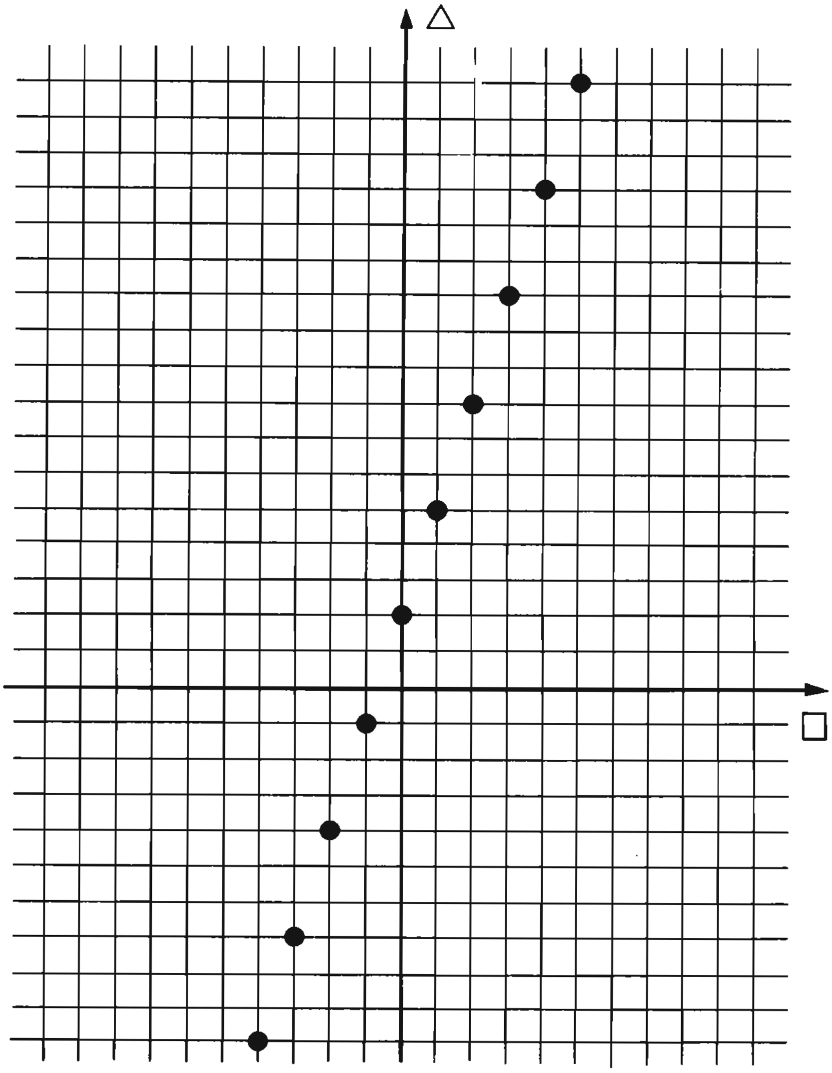
$5 \rightarrow \triangle$

$$\overset{5}{\triangle} = (3 \times \boxed{1}) + {}^{+}2$$

$$5 = 3 + {}^{+}2$$

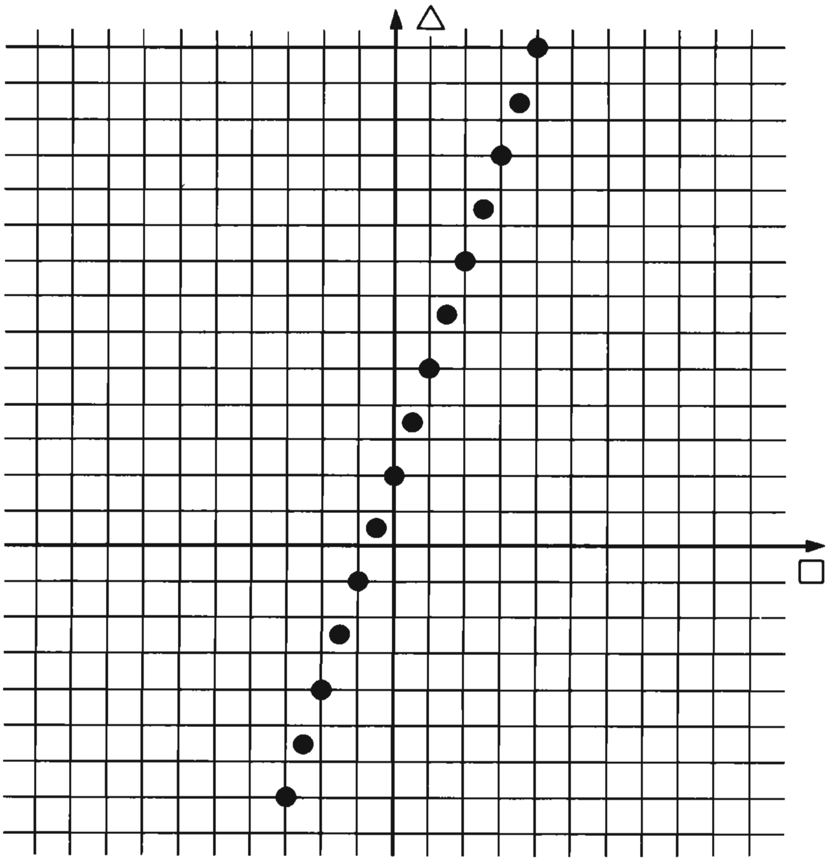
(12) Can you mark 3 more points on Phil's graph **without** doing the arithmetic?

(12) Here are some more points (using only whole numbers) on Phil's graph. There are many others, but all *correct* points exhibit this same pattern.



You will need to rely on your own judgment as to whether or not to use fractions at this stage. It is recommended that fractions be used in a later lesson, *after* the children have all discovered the *slope* idea.

If you *do* use fractions, your graph will look more like this:



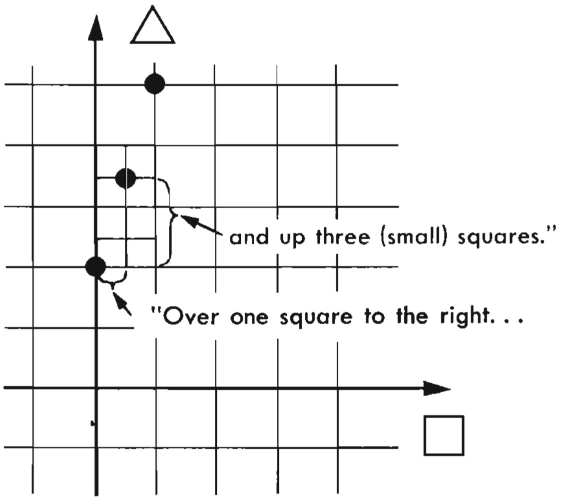
One possible graph showing truth set of

$$\triangle = (3 \times \square) + {}^{+}2,$$

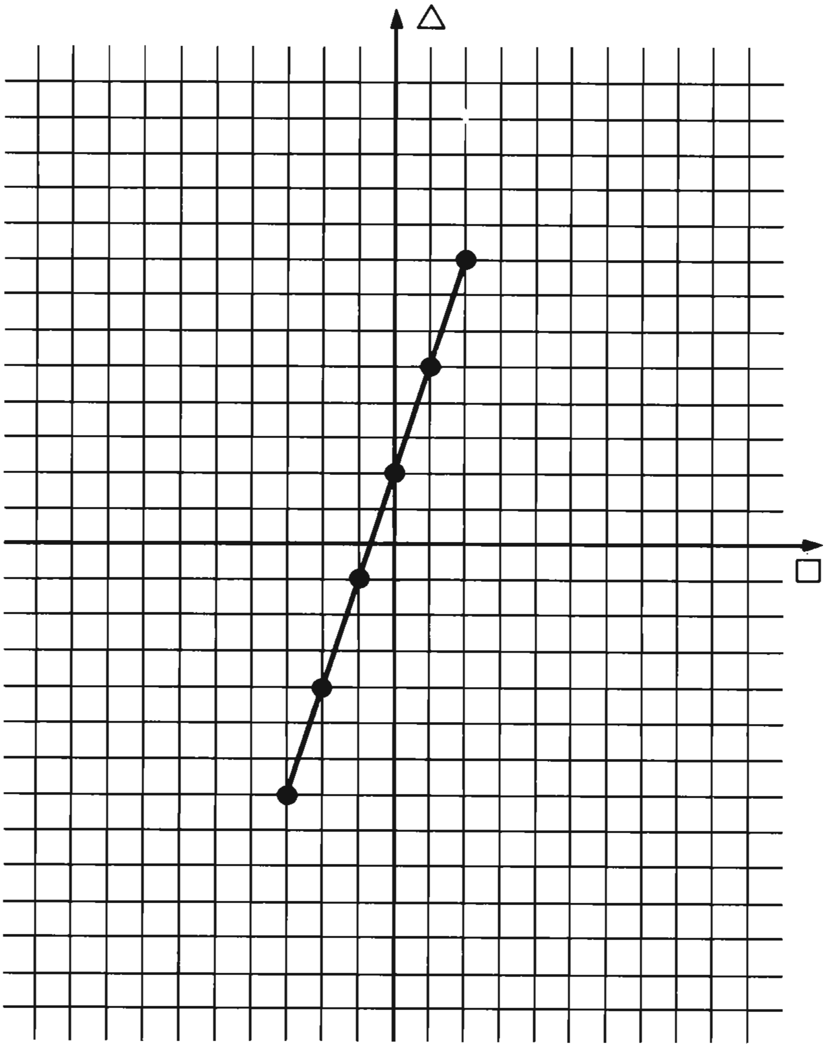
using whole numbers and also fractions
(not recommended at this stage).

The pattern, of course, is still "over one and up three," but when you use fractions, you can count with squares of any size you want. For example, when counting with small squares this size,

the pattern "over one, up three" looks like this:



If you filled in *all* of the *possible* fractions, the truth set would look like this:



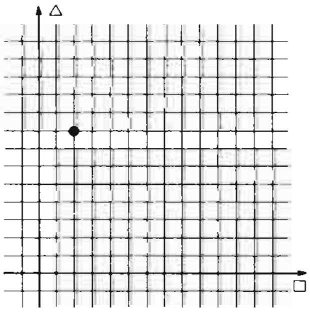
(13) Check your points by substituting into the equation. Were your points right?

[page 20]

(14) Claudia started a graph for the equation

$$\triangle = (5 \times \square) + {}^{-}2$$

this way:



Does Claudia's point work?

(13) Check exactly as in the earlier examples.

(14) Claudia's point does work. It has co-ordinates (2, 8).

$2 \rightarrow \square$

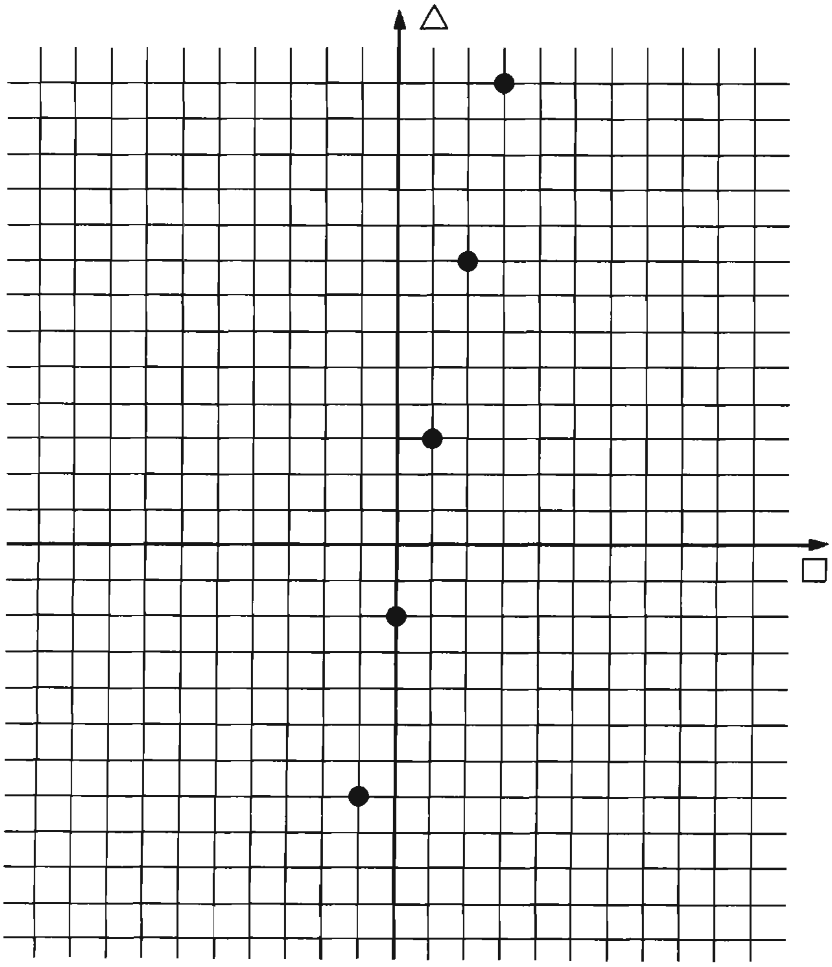
$8 \rightarrow \triangle$

$$\overset{8}{\triangle} = (5 \times \boxed{2}) + {}^{-}2$$

$$8 = 10 + {}^{-}2 \qquad \text{True}$$

(15) Can you mark two more points on Claudia's graph without doing the arithmetic?

(15) Here are some more points on Claudia's graph. The points shown here are $(^-1, ^-7)$, $(0, ^-2)$, $(1, 3)$, $(2, 8)$, $(3, 13)$:



(16) Substitute into the open sentence. Did you get a true statement?

(16) Follow the same pattern as in earlier questions.

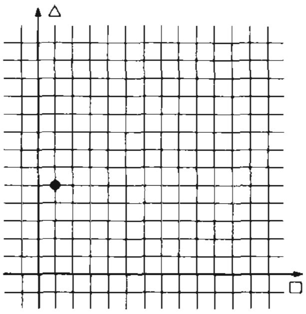
(17) Were your points right?

(17) If the open sentence became *true*, the points were right; if it became *false*, they were not right.

(18) Jill started a graph for

$$\triangle = (2 \times \square) + 3$$

like this:



Do you agree?

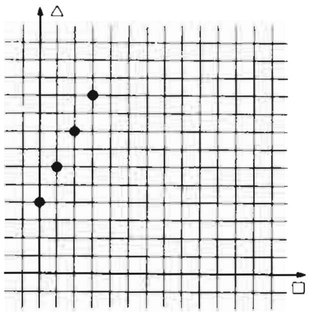
(18) Jill's point, (1, 5), is right for the truth set of the equation $\triangle = (2 \times \square) + 3$.

(19) Can you mark 3 more points on Jill's graph without doing the arithmetic?

(20) Now substitute into the equation. Were your points right?

[page 21]

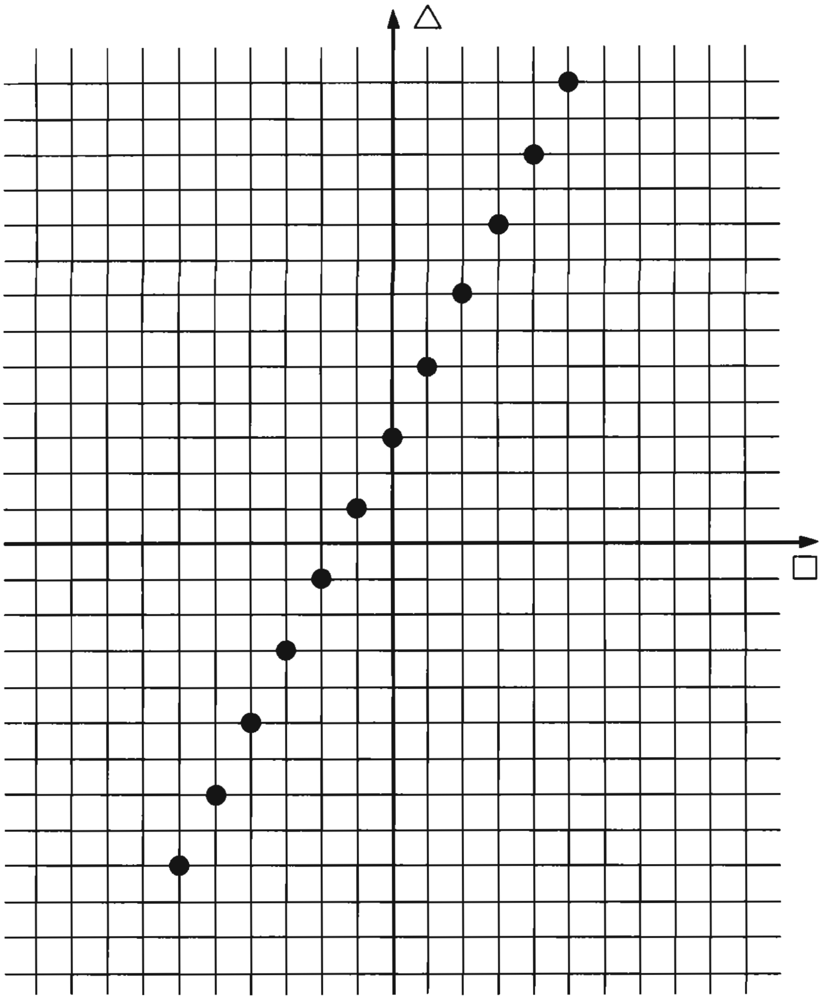
(21) Lou made this graph.



Can you write the equation that Lou was using?

$$\triangle = (\underset{?}{\underset{\uparrow}{\quad}} \times \square) + {}^{+}4$$

(19) Here are some more points:



The idea in question 19, of course, is to locate points by using the pattern "over one square to the right and up two squares," rather than by substituting into the equation.

(20) If the open sentence became *true*, the points were right; if it became *false*, they were not right.

(21) Since the pattern on Lou's graph is "over one to the right and up two," the equation must be:

$$\triangle = ({}^{+}2 \times \square) + {}^{+}4$$

chapter 12 / *Pages 21–23 of Student Discussion Guide*

POSTMAN STORIES

The idea of this chapter is to be able to make up a postman story corresponding to a problem, or to make up a problem corresponding to a postman story.

For example: *Given* this postman story, "A postman brings you a check for \$3, and then he brings you a bill for \$2 . . . ," you can make up this numerical problem, $^{+}3 + {}^{-}2 =$ ______.

Or, given this problem, $^{+}10 + {}^{-}3 =$ ______, you can make up this corresponding postman story, "The postman brings you a check for \$10, and he also brings you a bill for \$3 . . ."

There is a definite procedure for matching postman stories with numerical problems, and vice versa, in this way:

(a) A *check* is represented by a positive number.
(b) A *bill* is represented by a *negative* number.
(c) Something *brought to you* is represented by the addition sign +.
(d) Something *taken away* from you is represented by the *subtraction* sign −.
(e) In a sum, something happens *and then* something else *also* happens.
(f) In a product, the second factor is a *bill* or a *check;* the first factor is *how many* bills or *how many* checks.

It is better not to tell these rules to the children, except perhaps casually and somewhat unobtrusively.

Chapter 12
POSTMAN STORIES [**page 21**]

We would like to invent an arithmetic for numbers with signs.

To do this, we look at an example.

Suppose a postman brings you a check for \$3. We can represent this as $^{+}3$. If he brings you a **bill** for \$2, we can represent that as $^{-}2$.

(1) Suppose the postman brings you a check for \$5 and a bill for \$3. Are you richer or poorer? By how much?

Can you make up a postman story for each problem? What answer do you get for each problem?

(2) $^{+}2 + {}^{+}4 = ?$

ANSWERS AND COMMENTS

(1) **Richer; by \$2.**

(2) **The postman brings you a check for \$2, and he also brings you a check for \$4. As a result, you are richer by an amount of \$6; $^{+}6$.**

(3) $^{+}5 + {}^{-}2 = ?$

(3) The postman brings you a check for \$5, and he also brings you a bill for \$2. As a result, you are richer by the amount of \$3; $^{+}3$.

(4) $^{-}2 + {}^{-}3 = ?$

(4) The postman brings you a bill for \$2, and he also brings you a bill for \$3. As a result, you are poorer by the amount of \$5; $^{-}5$.

(5) $^{+}5 + {}^{-}6 = ?$

(5) The postman brings you a check for \$5, and he also brings you a bill for \$6. As a result, you are poorer by the amount of \$1; $^{-}1$.

(6) $^{-}7 + {}^{+}9 = ?$

(6) The postman brings you a bill for \$7, and he also brings you a check for \$9. As a result, you are richer by the amount of \$2; $^{+}2$.

(7) $^{-}5 + {}^{+}1 = ?$

(7) The postman brings you a bill for \$5, and he also brings you a check for \$1. As a result, you are poorer by the amount of \$4; $^{-}4$.

(8) $^{-}3 + 0 = ?$

(8) The postman brings you a bill for \$3, and he also brings you a picture postcard (or something else involving no gain or loss of money). As a result, you are poorer by the amount of \$3; $^{-}3$.

Of course, this is a special case of the identity

$$\square + 0 = \square,$$

which is known as the *addition law for zero.*

[page 22]

(9) $^{-}2 + {}^{-}5 = ?$

(9) The postman brings you a bill for \$2 and another bill for \$5. You are poorer by \$7; $^{-}7$.

(10) $^{+}6 + {}^{+}3 = ?$

(10) The postman brings you a check for \$6 and a check for \$3. You are richer by the amount of \$9; $^{+}9$.

(11) $^{+}8 + {}^{-}1 = ?$

(11) The postman brings you a check for \$8 and a bill for \$1. You are richer by an amount of \$7; $^{+}7$.

(12) $^{-}3 + {}^{-}6 = ?$

(12) The postman brings you a bill for \$3, and he also brings you a bill for \$6. You are poorer by the amount of \$9; $^{-}9$.

(13) $^{+}1 + {}^{+}12 = ?$

(13) The postman brings you a check for \$1 and a check for \$12. You are \$13 richer; $^{+}13$.

(14) $^{+}7 + {}^{-}9 = ?$

(14) The postman brings you a check for \$7 and a bill for \$9. You are \$2 poorer; $^{-}2$.

(15) $^{+}2 + {}^{+}17 = ?$

(15) The postman brings you checks for \$2 and \$17. This makes you richer by \$19; $^{+}19$.

(16) See if you can make up a story that will correspond to **subtraction?** How would you explain this?

$$^{+}7 - {}^{+}2 = ?$$

(16) The postman *brings* you a check for \$7, and he *takes away* a check for \$2. As a result of that visit, he has made you richer by the amount of \$5; $^{+}5$.

If a postman
brings you a **check** for \$3,
we can write this $^+3$;
or if he **brings** a **bill** for \$3,
we can write this $^-3$;
or if he
takes away a **check** for \$3,
we can write this $- {}^+3$;
or if he
takes away a **bill** for \$3,
we can write this $- {}^-3$.

Subtraction problems inevitably involve some concept of "undoing." You can have the postman *take away* bills and checks instead of bringing them. Another way is to make up "Billy-and-I" stories using money and I. O. U.'s. Whenever I borrow money from Billy, I give him an I. O. U., and vice versa. To make up "Billy-and-I" stories, you can rummage around in your closet and drawers. You can *find* money; you can *find* I. O. U.'s; Billy can *find* I. O. U.'s; you and Billy can *lose* I. O. U.'s; you can *lose* money; and so on. If *finding* represents *addition*, then *losing* represents *subtraction*.

(17) If the postman **brings** you a **check** for \$5, does his visit make you **richer** or **poorer?**

(17) **Richer**

(18) If the postman **brings** you a **bill** for \$5, does his visit make you **richer** or **poorer?**

(18) **Poorer**

(19) The postman comes on Monday. He says, "I'm sorry, the check I left last week was really for the man next door." He takes back the check (which happened to be for \$21). Does his visit on Monday make you richer or poorer?

(19) **Poorer**

Of course, the postman's visit *last week* made you *richer;* and the *two visits together* simply cancel out and make you neither richer nor poorer:

$$^+21 - {}^+21 = 0.$$

Sometimes the children confuse one of these questions with one of the others. Question 19 asked *only* about the postman's visit on *this Monday.* The answer, therefore, is that his visit *this Monday* made you \$21 poorer.

(20) The postman comes on Thursday. He says, "I'm sorry. That bill that I brought yesterday was really for the family upstairs. I do hope you didn't worry about it."

He takes away the bill (which was for \$100).

Did his visit on Thursday make you **richer** or **poorer?**

(20) **Richer**

(21) Bill made up this story for

$$^+2 + {}^+3 = \underline{\quad}.$$

Bill said: "The postman brought

you a check	$\overset{+}{\downarrow}$
for \$2	$^+2$
and	$^+2 \underset{\uparrow}{+}$
at the same time he brought you another check	$^+2 + \overset{\downarrow}{^+}$
for \$3.	$^+2 + \overset{\downarrow}{^+3} = \underline{\quad}$
As a result of his visit, you were richer by \$5."	$^+2 + {}^+3 = \overset{\downarrow}{^+5}$

Is Bill's story right or wrong?

[page 23]

(21) **Bill's story is a good one.**

(22) Janet made up this story for $^{+}5 - ^{+}7 = \_\_\_.$ She said: "The postman brought you a check $\overset{\downarrow}{^{+}}$ for \$5 $\overset{\downarrow}{^{+}5}$ but he also took away $^{+}5 \overset{\downarrow}{-}$ a check $^{+}5 - \overset{\downarrow}{^{+}}$ for \$7." $^{+}5 - \overset{\downarrow}{^{+}7} = \_\_\_$ As a result of **this** visit, are you **richer** or **poorer?**	**(22) Poorer, by \$2.**
(23) What answer do you get for $^{+}5 - ^{+}7 = \_\_\_?$	**(23) $^{-}2$**

Can you make up a postman story for each example? What answers do you get?

(24) $^{+}10 + ^{+}2 = ?$	**(24) Brings check for \$10; brings check for \$2; $^{+}12$**
(25) $^{-}5 + ^{-}3 = ?$	**(25) Brings bill for \$5; brings bill for \$3; $^{-}8$**
(26) $^{+}5 - ^{+}2 = ?$	**(26) Brings check for \$5; takes away check for \$2; $^{+}3$**
(27) $^{+}7 - ^{+}3 = ?$	**(27) Brings check for \$7; takes away check for \$3; $^{+}4$**
(28) $^{+}15 + ^{-}5 = ?$	**(28) Brings check for \$15; brings bill for \$5; $^{+}10$**
(29) $^{+}3 + ^{+}8 = ?$	**(29) Brings check for \$3; brings bill for \$8; $^{+}11$**
(30) $^{-}20 + ^{-}5 = ?$	**(30) Brings bill for \$20; brings bill for \$5; $^{-}25$**
(31) $^{+}20 + ^{-}5 = ?$	**(31) Brings check for \$20; brings bill for \$5; $^{+}15$**
(32) $^{+}20 - ^{+}7 = ?$	**(32) Brings check for \$20; takes away check for \$7; $^{+}13$**
(33) $^{+}10 - ^{-}3 = ?$	**(33) Brings check for \$10; takes away bill for \$3; $^{+}13$**
(34) $^{-}5 + ^{-}7 = ?$	**(34) Brings bill for \$5; brings bill for \$7; $^{-}12$**

chapter 13 / *Page 24 of Student Discussion Guide*

MORE OPEN SENTENCES

There will probably be nothing new in this chapter for most of the students in your class.

Perhaps all your children have not yet discovered the secrets of quadratic equations; this will give them another chance. In any event, this will give *all* the children some more *experience* with algebraic ideas.

Chapter 13
MORE OPEN SENTENCES

[**page 24**]

Can you find the truth set for each open sentence?

ANSWERS AND COMMENTS

(1)	$(\square \times \square) - (13 \times \square) + 30 = 0$	(1)	**{3, 10} (which, of course, is the same as {10, 3}, since this is a *set* and not an ordered pair)**
(2)	$(\square \times \square) - (107 \times \square) + 700 = 0$	(2)	**{7, 100}**
(3)	$(\square \times \square) - (29 \times \square) + 100 = 0$	(3)	**{25, 4}**
(4)	$(\square \times \square) - (5 \times \square) + 6 = 0$	(4)	**{2, 3}**
(5)	$(\square \times \square) - (7 \times \square) + 12 = 0$	(5)	**{3, 4}**
(6)	$(\square \times \square) - (53 \times \square) + 150 = 0$	(6)	**{50, 3}**
(7)	$(\square \times \square) - (23 \times \square) + 42 = 0$	(7)	**{21, 2}**
(8)	$(\square \times \square) - (18 \times \square) + 77 = 0$	(8)	**{7, 11}**
(9)	$(\square \times \square) - (4 \times \square) + 3 = 0$	(9)	**{3, 1}**
(10)	$(\square \times \square) - (104 \times \square) + 303 = 0$	(10)	**{101, 3}**
(11)	$(\square \times \square) - (12 \times \square) + 35 = 0$	(11)	**{7, 5}**
(12)	$(\square \times \square) - (793 \times \square) + 792 = 0$	(12)	**{792, 1}**
(13)	$(\square \times \square) - (34 \times \square) + 93 = 0$	(13)	**{3, 31}**
(14)	$(\square \times \square) - (16 \times \square) + 55 = 0$	(14)	**{5, 11}**
(15)	$(\square \times \square) - (30 \times \square) + 189 = 0$	(15)	**{9, 21}**

Problem 15 is probably the most difficult in this group. You may want to attack it by means of prime factors.

You could, for example, proceed as follows.

Say:	Write on board:
Three is a factor of 189 (which I just happened to notice by inspection).	$\begin{array}{r} 69 \\ 3\overline{)189} \end{array}$

Say:	Write on board:
Therefore, *one* factorization of 189 is . . .	$3 \times 62 = 189$
Three plus 63 is not equal to 30, so this doesn't work.	$3 + 63 \neq 30$ (erase)
Three is a factor of 63.	$\begin{array}{r} 21 \\ 3\overline{)63} \end{array}$
In fact, here is another factorization of 189 . . .	$(3 \times 3) \times 21 = 189$
or . . .	$9 \times 21 = 189$
Now let's see if this works . . .	$9 + 21 = 30$
It does! So the answer is . . .	$\{9, 21\}$

(16) $^{+}30 + \square = {}^{+}37$

(16) $\{^{+}\mathbf{7}\}$

(17) $^{+}30 + \square = {}^{+}20$

(17) $\{^{-}\mathbf{10}\}$

If the children suggest $^{+}10$, write

$$^{+}30 + {}^{+}10 = ?$$

and ask them for the sum. They will surely say $^{+}40$, and this will probably provide the necessary insight for them to solve the original problem.

(18) $(^{+}2 \times \square) + {}^{+}11 = {}^{+}51$

(18) $\{\mathbf{20}\}$ **or** $\{^{+}\mathbf{20}\}$

(19) $(^{+}2 \times \square) + {}^{-}10 = {}^{+}16$

(19) $\{^{+}\mathbf{13}\}$

(20) $(^{+}2 \times \square) + {}^{+}6 = 0$

(20) $\{^{-}\mathbf{3}\}$

If the children say $^{+}3$, write

$$(^{+}2 \times {}^{+}3) + {}^{+}6 = ?$$

and handle the problem in a similar way to problem 17.

(21) $(5 \times \square) + 6 = 46$

(21) $\{\mathbf{8}\}$

chapter 14 / *Page 25 of Student Discussion Guide*

MORE NUMBERS WITH SIGNS

This chapter has been presented experimentally very early in the year's work—in the second or third lesson. This seems to work rather well. You may even find that you get better results if you teach multiplication of signed numbers *before* you teach addition of signed numbers. The decision is yours.

Chapter 14

MORE NUMBERS WITH SIGNS [page 25]

How can we make up a postman story for

$$^{+}2 \times {}^{+}3?$$

Let's agree to call the second factor a bill or a check, $\_\_\_ \times \underset{\uparrow}{{}^{+}3}$

and we'll call the first factor the **number** of checks or bills. $\underset{\uparrow}{{}^{+}2} \times \_\_\_$

Can you make up a postman story for each problem? What answer do you get for each problem?

ANSWERS AND COMMENTS

(1) $^{+}2 \times {}^{+}3 = ?$	(1) **The postman *brings*** $\overset{+}{\uparrow}$ **you *two*** $\underset{\uparrow}{{}^{+}2} \times$ ***checks*** $^{+}2 \times \overset{+}{\uparrow}$ **for \$3** $^{+}2 \times \underset{\uparrow}{{}^{+}3} =$ **so you are *richer*** $^{+}2 \times {}^{+}3 = \overset{+}{\uparrow}$ **by the amount of \$6.** $^{+}2 \times {}^{+}3 = \underset{\uparrow}{{}^{+}6}$
(2) $^{+}5 \times {}^{+}2 = ?$	(2) **Brings five checks for \$2; richer by \$10; $^{+}10$**
(3) $^{+}2 \times {}^{-}3 = ?$	(3) **Brings two bills for \$3; poorer by \$6; $^{-}6$**
(4) $^{+}2 \times {}^{-}10 = ?$	(4) **Brings two bills for \$10 each; poorer by \$20; $^{-}20$**
(5) $^{+}3 \times {}^{-}4 = ?$	(5) **Brings three bills for \$4 each; poorer by \$12; $^{-}12$**
(6) $^{+}5 \times {}^{+}7 = ?$	(6) **Brings five checks, each for \$7; richer by \$35; $^{+}35$**
(7) $^{-}2 \times {}^{-}3 = ?$	(7) **Takes away two bills for \$3 each; richer by \$6; $^{+}6$**
(8) $^{+}3 \times {}^{+}10 = ?$	(8) **Brings three checks, for \$10 each; richer by \$30; $^{+}30$**
(9) $^{-}2 \times {}^{-}100 = ?$	(9) **Takes away two bills for \$100 each; richer by \$200; $^{+}200$**

It may help to ask the children these questions: If the postman *brings* a check, are you richer or poorer? What if he *takes* *away* a check? If the postman *brings* a *bill*, are you *richer* or *poorer?* What if he *takes away* a bill?

(10) Cynthia made up this story:
The postman came. He
was handling **bills** ___ $\times$ ___
for $100 each. ___ $\times$ 100
How many bills? 2 $\times$ 100
Did he **bring** them, or
take them away? 2 $\times$ 100

As a result of this visit, were we **richer** or **poorer?**

(10) *Two* bills. He *took them away*. He makes us richer (if he brings bills, he makes us poorer; so, if he takes *away* bills we become *richer*).

(11) What answer do you get for

$$2 \times 100 = \_\_\_?$$

(11) **$^{+}200$**

chapter 15 / *Pages 26–28 of Student Discussion Guide*

Myrna's Discoveries

This chapter continues the discovery sequence on "slope" and "intercept" for straight-line graphs. This means, of course, the role played by the numbers ${}^{+}3$ and ${}^{-}2$ in an equation of this form:

$$\triangle = (\underset{\uparrow}{{}^{+}3} \times \square) + \underset{\uparrow}{{}^{-}2}.$$

The number here

$$\triangle = (\underset{\uparrow}{{}^{+}3} \times \square) + \underline{\hspace{2em}}$$

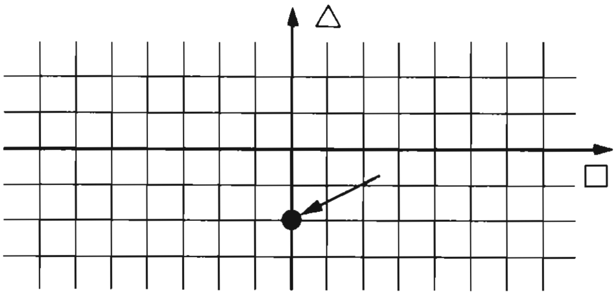
determines the pattern. The ${}^{+}3$ indicates a pattern of "over one square to the right and up three squares."

The number here

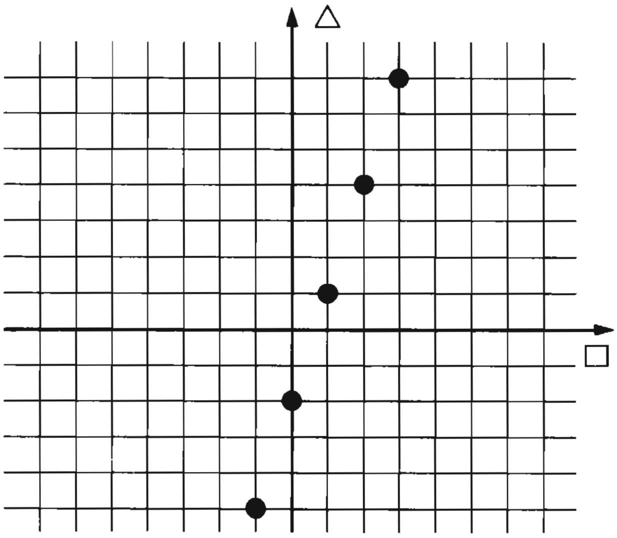
$$\triangle = (\underline{\hspace{2em}} \times \square) + \underset{\uparrow}{{}^{-}2}$$

indicates where the truth set intersects the vertical axis (labeled $\triangle$ or y).

The ${}^{-}2$ indicates that the intersection is at ${}^{-}2$:



Here is the graph for $\triangle = ({}^{+}3 \times \square) + {}^{-}2$ (using integers only):

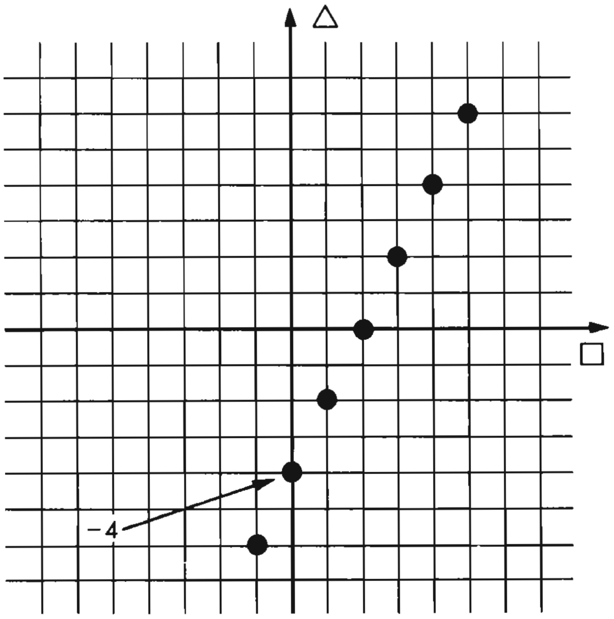


[page 26]

(1) Myrna says that she can look at a straight-line graph and tell what equation goes with it. Do you know how she does it?

ANSWERS AND COMMENTS

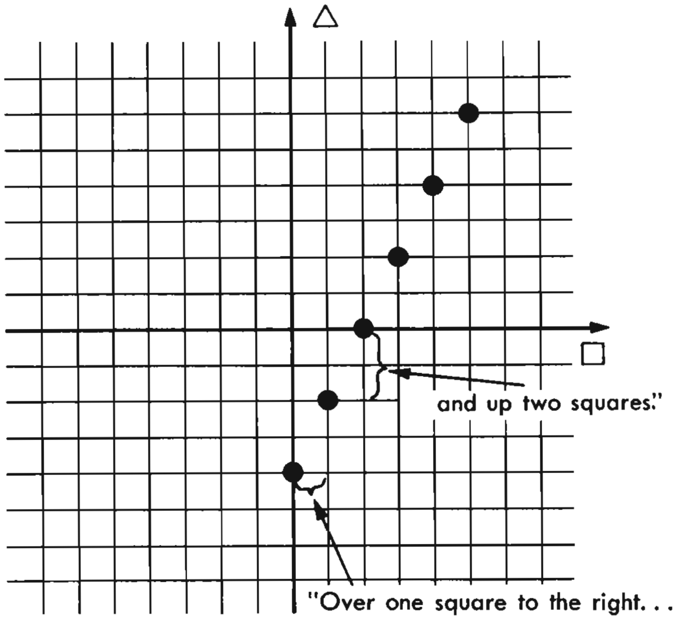
(1) Of course, Myrna looks at the *intersection* with the vertical axis:



This tells her the number that goes here:

$$\triangle = (\quad \times \square) + {}^{-}\underset{\uparrow}{4}.$$

Then she looks at the *pattern*:



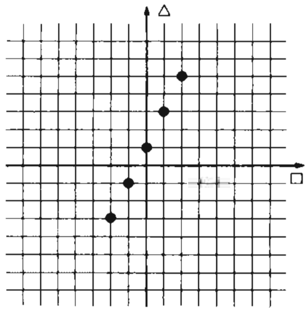
This tells her what number goes here:

$$\triangle = ({}^{+}\underset{\uparrow}{2} \times \square) + {}^{-}4.$$

Do not tell this to the children. Let the children discover it for themselves.

Of course, probably none of your children may have discovered these secrets as yet. The real purpose of this first question is to bring the matter to the children's attention, so they will realize that there *is* a secret to look for. Point out to the children that there *is* a "secret way" of matching graphs and equations. With this much of a hint they have no difficulty in finding the secret.

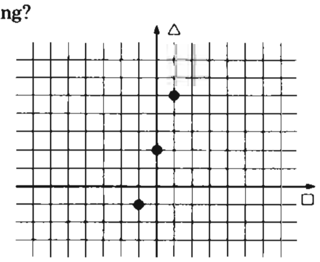
(2) Jerry made this graph. Can you tell what equation he was using?



$\triangle = (\_\_ \times \square) + \_\_$

(2) $\triangle = ({}^{+}2 \times \square) + {}^{+}1$

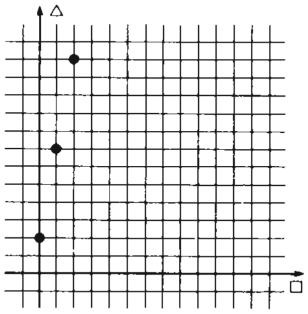
(3) Cecile made this graph. What equation was she using?



$\triangle = (\_\_ \times \square) + \_\_$

(3) $\triangle = ({}^{+}3 \times \square) + {}^{+}2$

(4) Lex made this graph. What equation was he using?

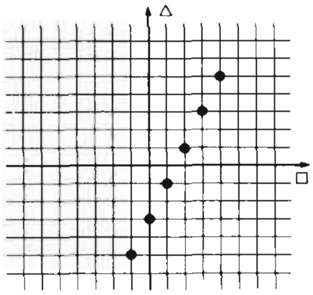


$\triangle = (\_\_ \times \square) + \_\_$

(4) $\triangle = ({}^{+}5 \times \square) + {}^{+}2$

[page 27]

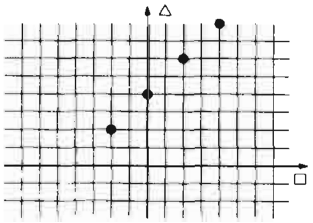
(5) Vivian made this graph. What equation did she use?



$\triangle = (\_\_ \times \square) + \_\_$

(5) $\triangle = ({}^{+}2 \times \square) + {}^{-}3$

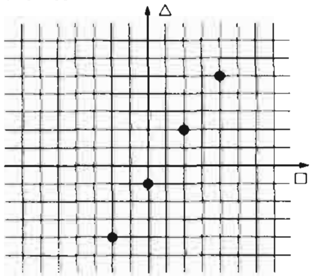
(6) Joe made this graph. Can you find the equation Joe was using?



$\triangle = (\_\_ \times \square) + \_\_$

(6) $\triangle = ({}^{+}1 \times \square) + {}^{+}4$

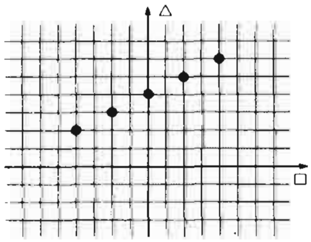
(7) Debbie made this graph. Do you know what equation she used?



$\triangle = (\_\_ \times \square) + \_\_$

(7) $\triangle = ({}^{+}\tfrac{3}{2} \times \square) + {}^{-}1$

(8) Bruce made this graph. What equation was he using?

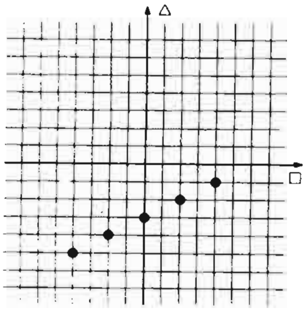


$\triangle = (\_\_ \times \square) + \_\_$

(8) $\triangle = (\tfrac{1}{2} \times \square) + {}^{+}4$

[page 28]

(9) Daria made this graph. What equation did she use?



$\triangle = (\_\_ \times \square) + \_\_$

(10) Do you know Myrna's two discoveries?

You may want to make up some graphs of your own and see if your friends can find the equations.

(9) $\triangle = \left({}^{+}\tfrac{1}{2} \times \square\right) + {}^{-}3$

(10) **This is a repeat of question 1—except that, by now, some of your children may know Myrna's secret method!**

chapter 16 / *Pages 28–29 of Student Discussion Guide*

HOW MANY BILLS?

This chapter uses postman stories for products only. It contains no examples involving addition or subtraction.

With some classes, the introduction of postman stories *only for products* has been tried. Also, these products were introduced quite early in the algebra lessons—prior to Chapter 16. The results have been very encouraging.

Consequently, you may want to use the contents of this chapter quite early in the year's work, and *before* you consider sums of signed numbers.

Chapter 16

HOW MANY BILLS?

[page 28]

(1) Can you make up the arithmetic problem for this postman story?

	${}^{\bigcirc}\underline{\quad} \times {}^{\bigcirc}\underline{\quad}$
The postman **brings**	${}^{\bigcirc}\underline{\quad} \times {}^{\bigcirc}\underline{\quad}$
three	${}^{\bigcirc}\underline{\quad} \times {}^{\bigcirc}\underline{\quad}$
bills	${}^{\bigcirc}\underline{\quad} \times {}^{\bigcirc}\underline{\quad}$
for $7 each.	${}^{\bigcirc}\underline{\quad} \times {}^{\bigcirc}\underline{\quad}$

As a result of this visit, are we richer or poorer? By how much? What answer do you get?

$${}^{\bigcirc}\underline{\quad} \times {}^{\bigcirc}\underline{\quad} = \underline{\quad}$$

(2) Can you make up the postman story for this problem?

$${}^{+}2 \times {}^{+}5 = \underline{\quad}$$

	The postman
$\nearrow{}^{\oplus}\underline{\quad} \times {}^{\bigcirc}\underline{\quad}$	brings takes away
$\underset{\uparrow}{{}^{+}2} \times {}^{\bigcirc}\underline{\quad}$	how many?
${}^{+}2 \times {}^{\oplus\swarrow}$	bills checks
${}^{+}2 \times \underset{\uparrow}{{}^{+}5}$	for ___ each
	so that we are
${}^{+}2 \times {}^{+}5 = {}^{\bigcirc\swarrow}\underset{\downarrow}{\underline{\quad}}$	richer poorer
${}^{+}2 \times {}^{+}5 = {}^{\bigcirc}\underline{\quad}$	by how much?

[page 29]

Can you make up a postman story for each problem? What answers do you get?

(3) $\quad {}^{+}5 \times {}^{+}10 = ?$

(4) $\quad {}^{+}10 \times {}^{-}3 = ?$

ANSWERS AND COMMENTS

(1) **Poorer; by $21; $^{+}3 \times {}^{-}7 = {}^{-}21$**

(2) **Brings; two; checks; $5; richer; $10**

(3) **The postman brings five checks for $10 each, so we are richer by $50; $^{+}50$**

(4) **Brings ten bills for $3 each; $^{-}30$**

(5) $^{+}2 \times {}^{-}1 = ?$

(5) Brings two bills for \$1 each; $^{-}2$

(6) $^{-}2 \times {}^{+}3 = ?$

(6) The postman takes away two checks for \$3 each; since *bringing* us checks makes us richer, *taking away* checks must make us poorer; $^{-}6$

(7) $^{-}3 \times {}^{+}1 = ?$

(7) Takes away three checks for \$1 each; $^{-}3$

You can also observe that this is a special case of the identity

$$\square \times 1 = \square.$$

(8) $^{-}1 \times {}^{+}7 = ?$

(8) Takes away one check for \$7; $^{-}7$

(9) $^{+}1 \times {}^{-}3 = ?$

(9) Brings one bill for \$3; $^{-}3$

(10) $^{-}1 \times {}^{-}5 = ?$

(10) The postman takes away one bill for \$5; since *bringing* bills makes us poorer, *taking away* bills must make us richer; $^{+}5$

You can also prove this by substituting into the *distributive law:*

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

$$\boxed{^{-}1} \times \left(\triangle_{^{-}5} + \nabla_{^{+}5}\right)$$

$$= \left(\boxed{^{-}1} \times \triangle_{^{-}5}\right) + \left(\boxed{^{-}1} \times \nabla_{^{+}5}\right)$$

$$^{-}1 \times 0 = ({}^{-}1 \times {}^{-}5) + ({}^{-}1 \times {}^{+}5)$$

$$0 = ({}^{-}1 \times {}^{-}5) + {}^{-}5$$

Consequently, the term $^{-}1 \times {}^{-}5$ must be the root of the equation $0 = \square + {}^{-}5$. In other words, $^{-}1 \times {}^{-}5 = {}^{+}5$.

(11) $^{-}2 \times {}^{+}15 = ?$

(11) Takes away two checks for \$15 each ;$^{-}30$

(12) $^{-}2 \times {}^{-}4 = ?$

(12) Takes away two bills for \$4 each; $^{+}8$

chapter 17 / *Page 29 of Student Discussion Guide*

MORE GRAPHS

This chapter extends the discussion of graphs to include:

(a) discrete negative slope (integers only),
(b) negative *y*-intercept (the second of Myrna's two discoveries).

At your own discretion you may wish to introduce the idea of substituting fractions into the box. You can do a little bit of this, rather unobtrusively, while carrying the main burden of the work with whole numbers only.

Chapter 17
MORE GRAPHS

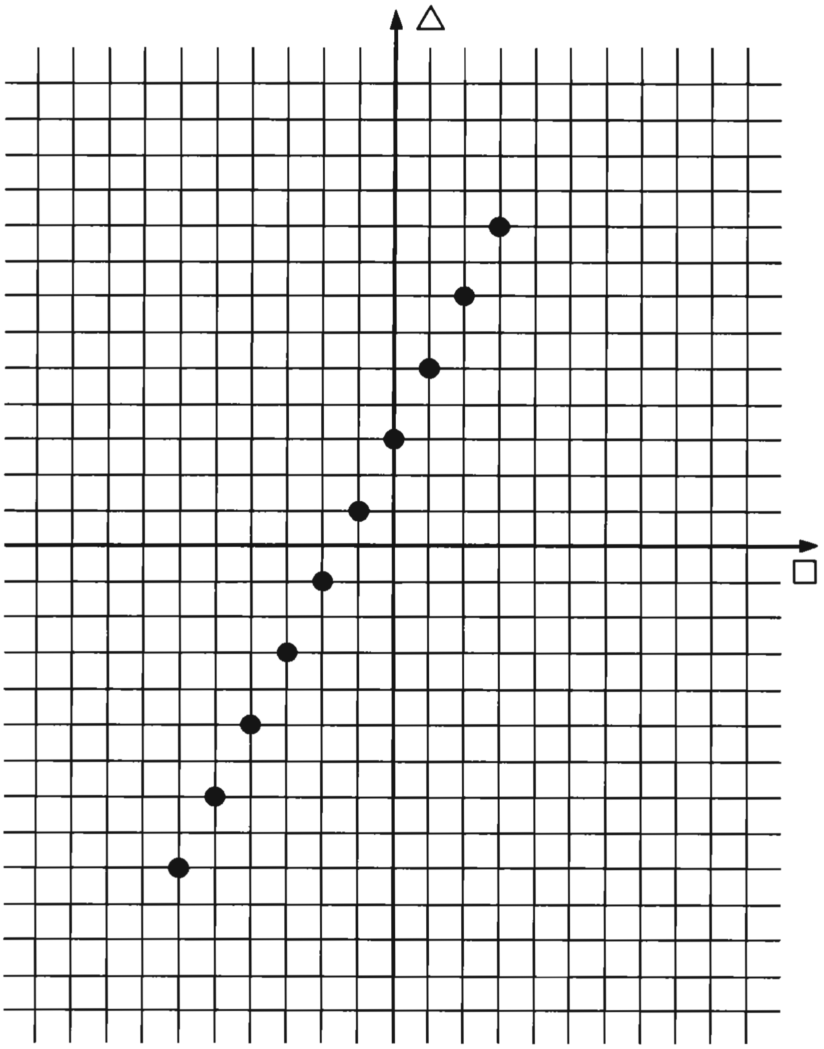
[page 29]

Can you use a graph to show each truth set?

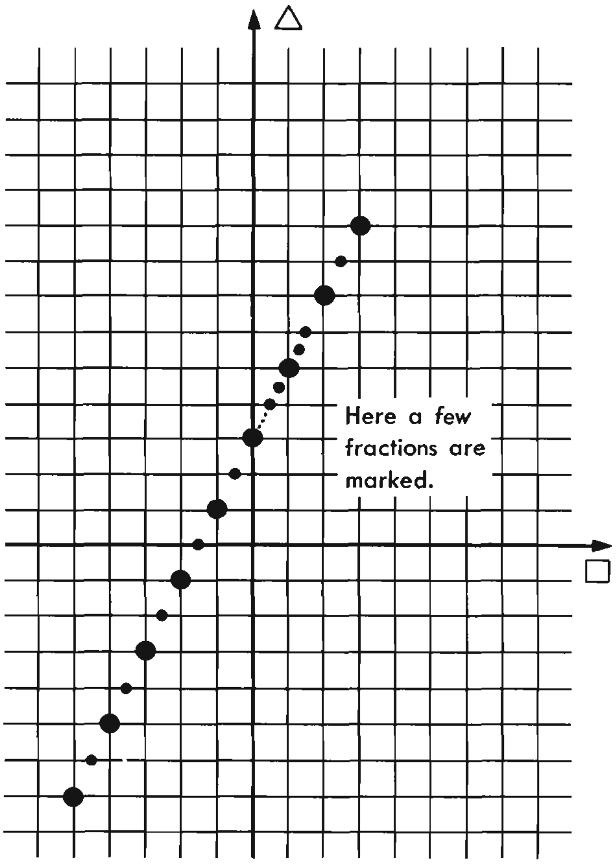
(1) $\triangle = ({}^{+}2 \times \square) + {}^{+}3$

ANSWERS AND COMMENTS

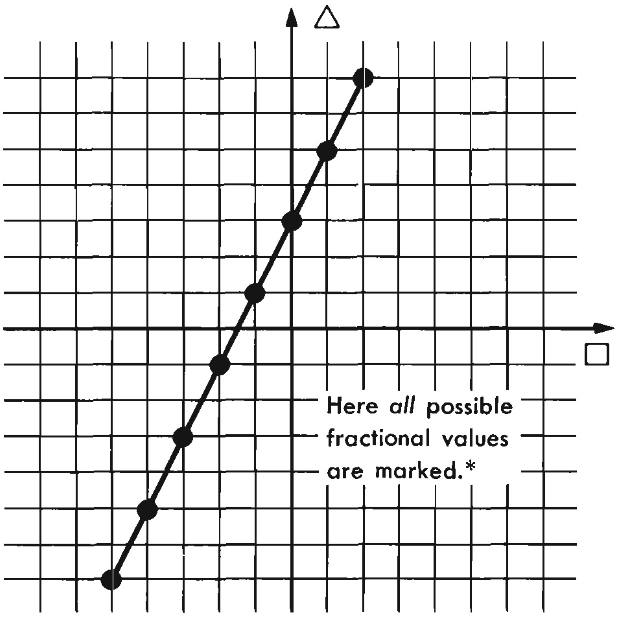
(1) **This is the picture using only whole numbers:**



Or if you substitute into the box using fractions and whole numbers both, the graph looks like this:



or even like this:



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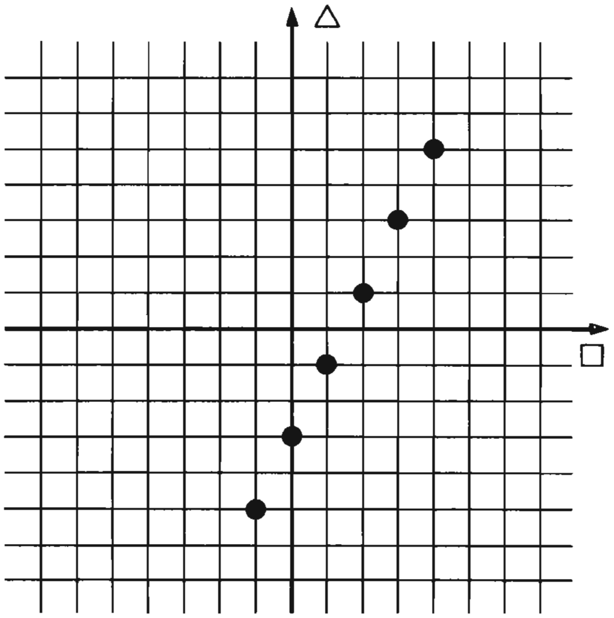
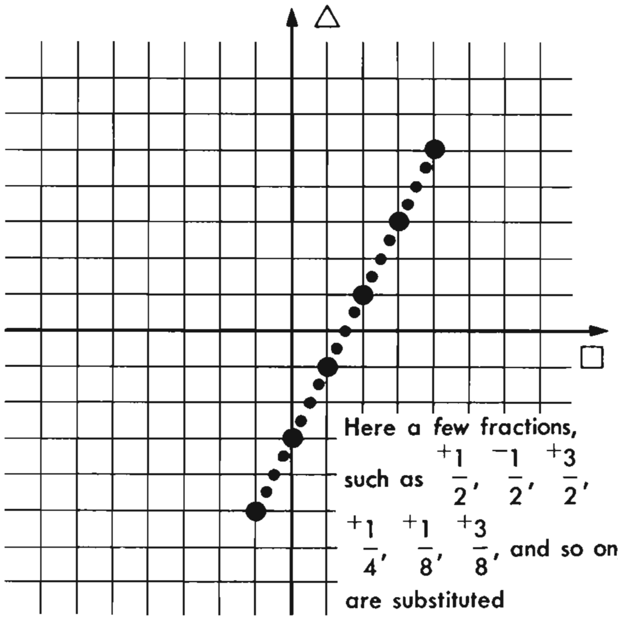
* There is a question of rational vs. irrational values of x (or box) involved here, but this is probably not the time to tell the children. Use your own judgment.

(2) $\triangle = (^{+}2 \times \square) + {}^{-}3$

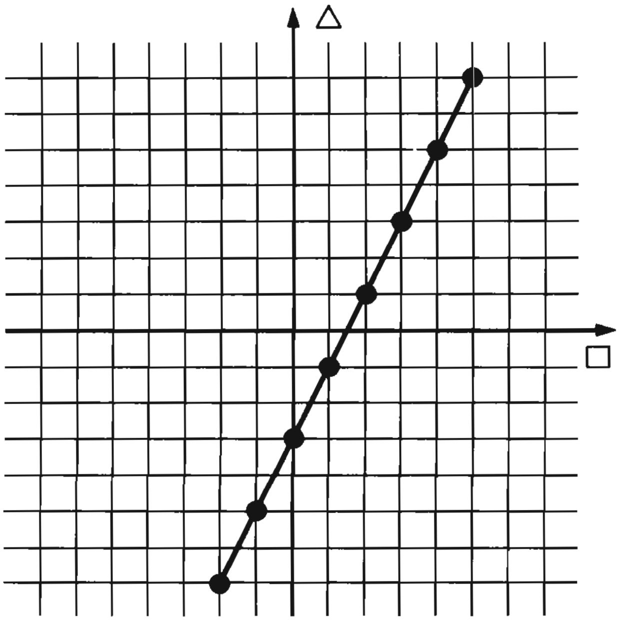
(2) **This is the picture if only whole numbers are substituted into the box:**



If you use both fractions and whole numbers to substitute into the box, the graph might look like this:

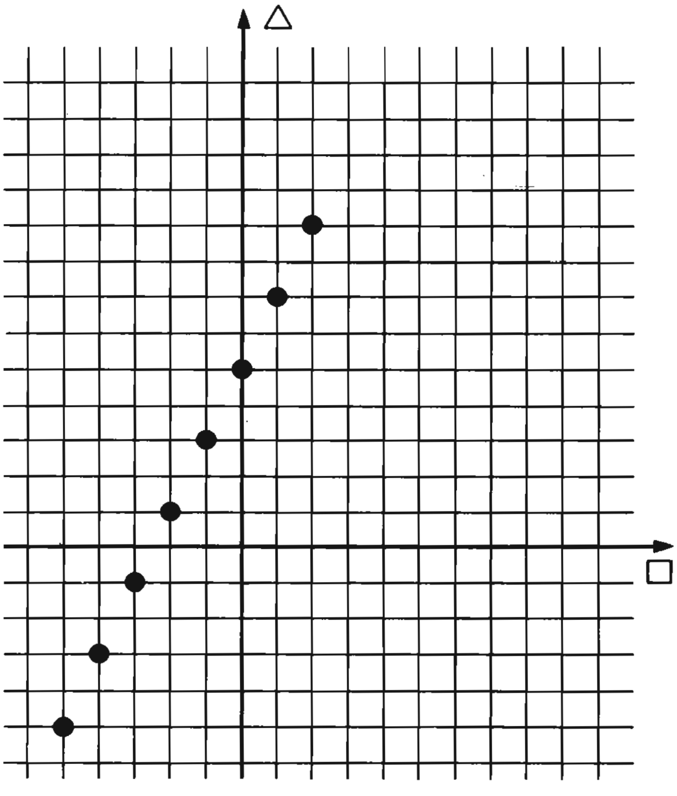


Here a few fractions, such as $\frac{^{+}1}{2}$, $\frac{^{-}1}{2}$, $\frac{^{+}3}{2}$, $\frac{^{+}1}{4}$, $\frac{^{+}1}{8}$, $\frac{^{+}3}{8}$, and so on, are substituted or even like this:

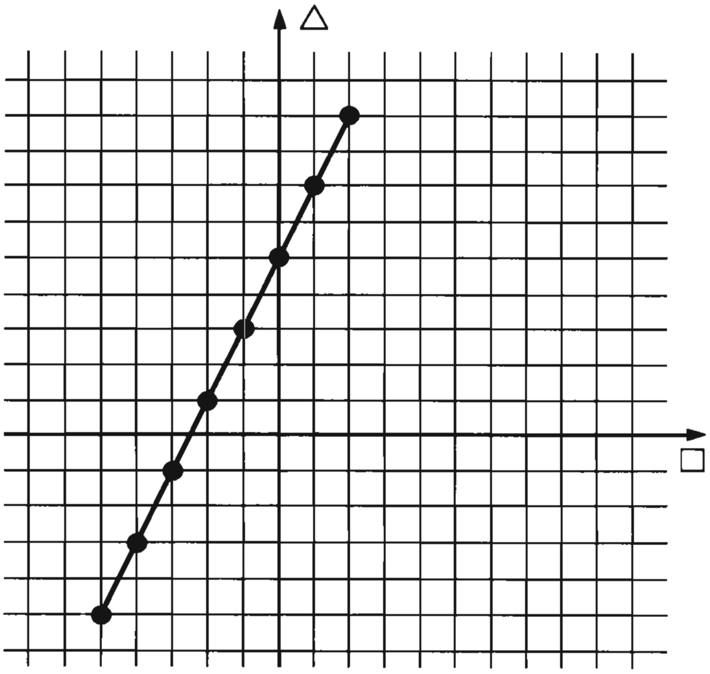


(3) $\triangle = (^{+}2 \times \square) + {}^{+}5$

(3) **The discrete graph (substituting only whole numbers $^{+}1$, $^{-}1$, $^{+}2$, $^{-}2$, etc., into the box) looks like this:**



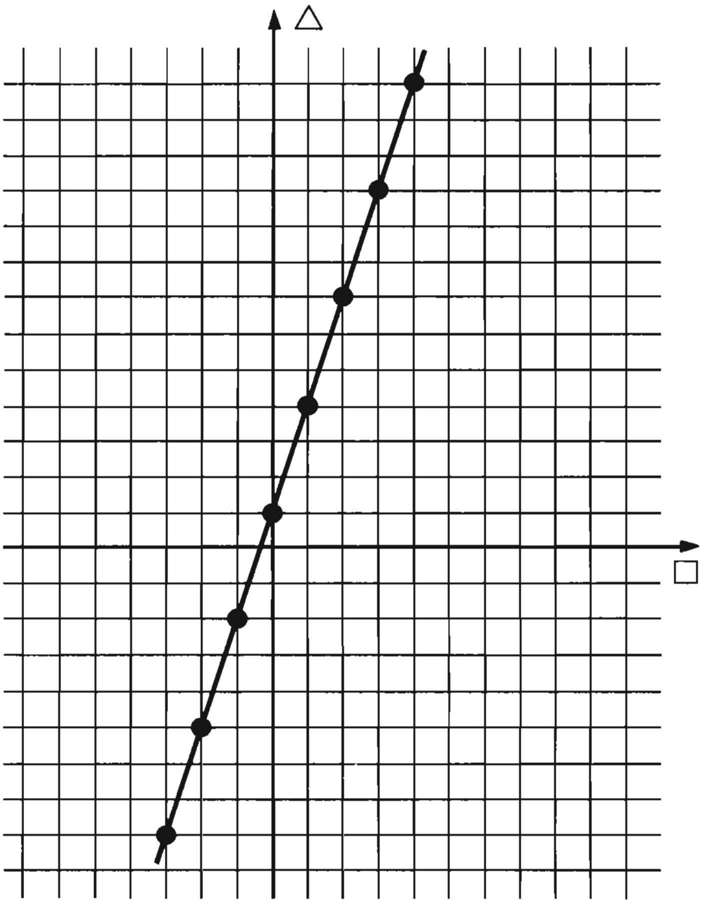
The *continuous* graph (allowing for all possible numbers—whole, fractional, or what have you—to be substituted into the box) looks like this:



For answers to questions 4 through 9, only the continuous graph is shown. You may prefer to work with your children using only discrete graphs (or, you may prefer to use some of each).

(4) $\triangle = ({}^{+}3 \times \square) + {}^{+}1$

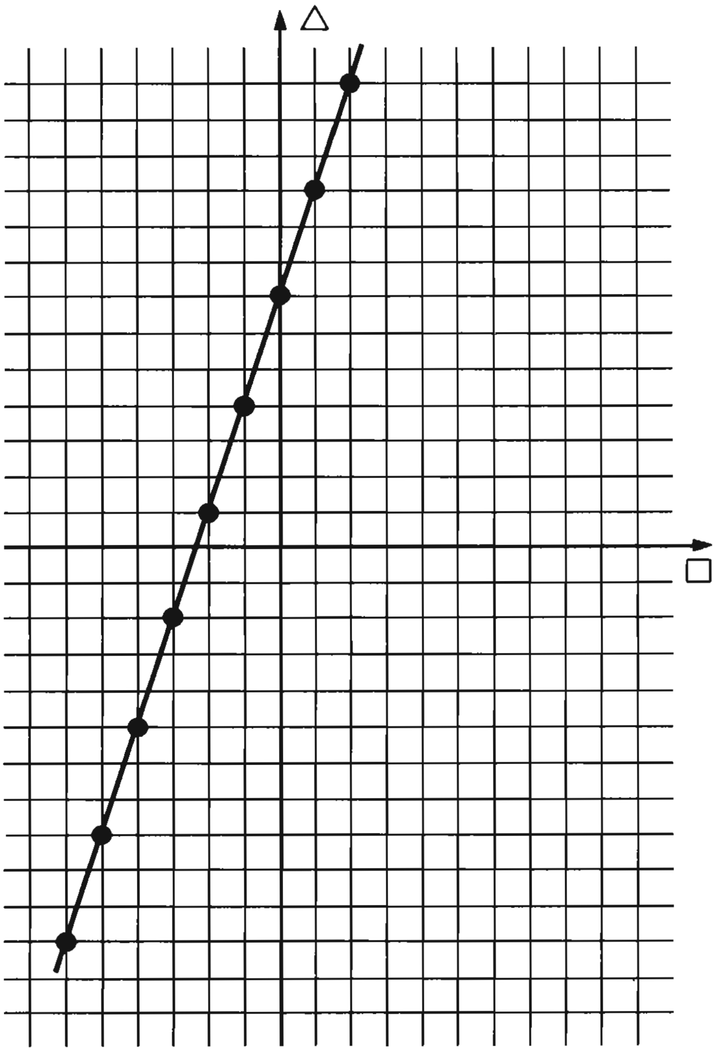
(4)



The slope pattern of the graph for question 4 is: over one square to the right and up three squares. The *y*-intercept is $^+1$. Of course, the slope pattern can be seen more easily from the discrete graph, where only whole numbers are substituted into the box.

(5) $\triangle = (^+3 \times \square) + ^+7$

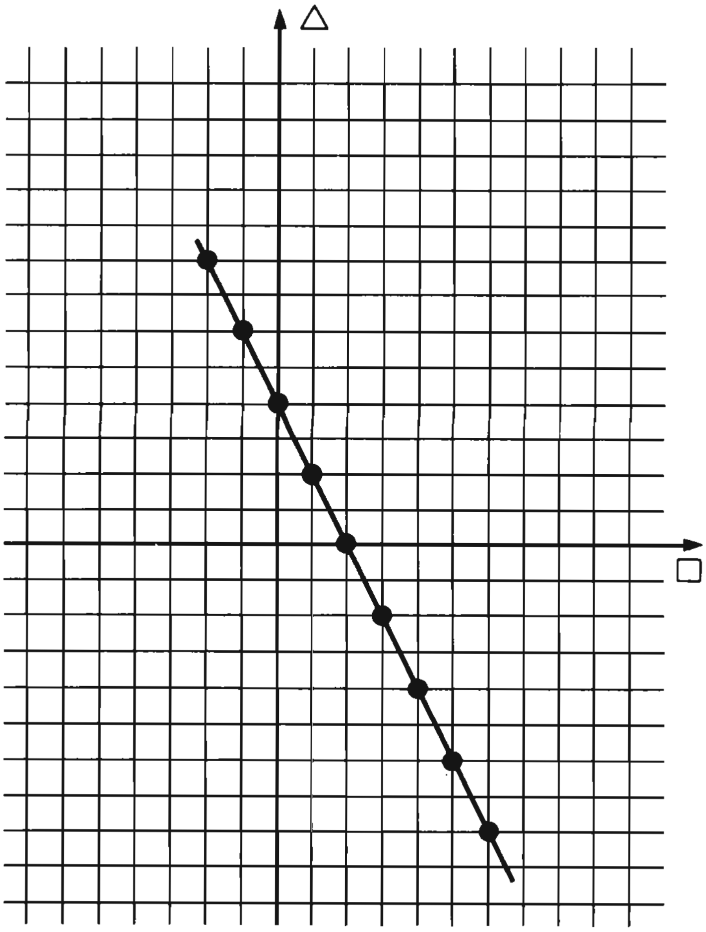
(5)



The slope pattern here is: over one square to the right and up three squares. The *y*-intercept is $^+7$. Again, you could see the slope pattern much more easily on the integer-only discrete graph than you can on a continuous graph.

(6) $\triangle = (^-2 \times \square) + ^+4$

(6)



The slope pattern is: over one square to the right and down two squares. The y-intercept is $^+4$.

Somewhere (perhaps about now) you may want to show your children the geometric meaning of y-intercept by substituting 0 into the box and comparing the geometrical meaning of substituting 0 into the box with the algebraic meaning of substituting 0 into the box.

Geometrically, substituting 0 into the box means locating a point somewhere on the vertical y-axis.

Algebraically, substituting 0 into the box means that this term

$$\triangle = \underbrace{(^-2 \times \boxed{0})}_{\uparrow} + ^+4$$

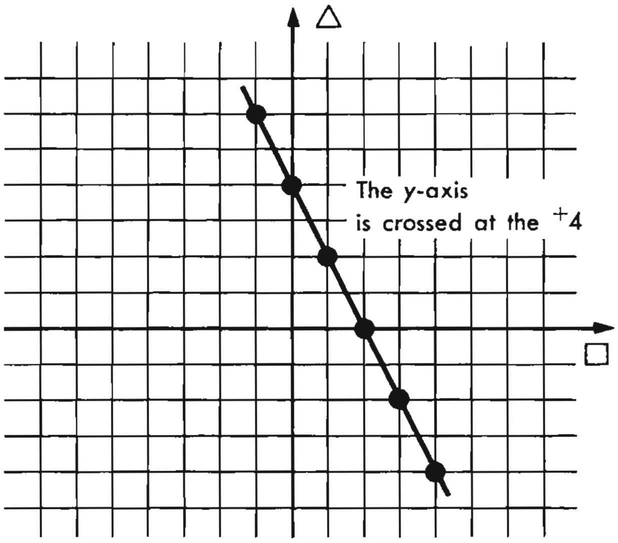
becomes 0 no matter what the slope coefficient may be, and we get the equation $\triangle = 0 + ^+4$. In order to make this *true*, we must substitute $^+4$ into $\triangle$. Combining these facts, we see the reason why the number *here*

$$\triangle = (^-2 \times \square) + \underset{\uparrow}{^+4}$$

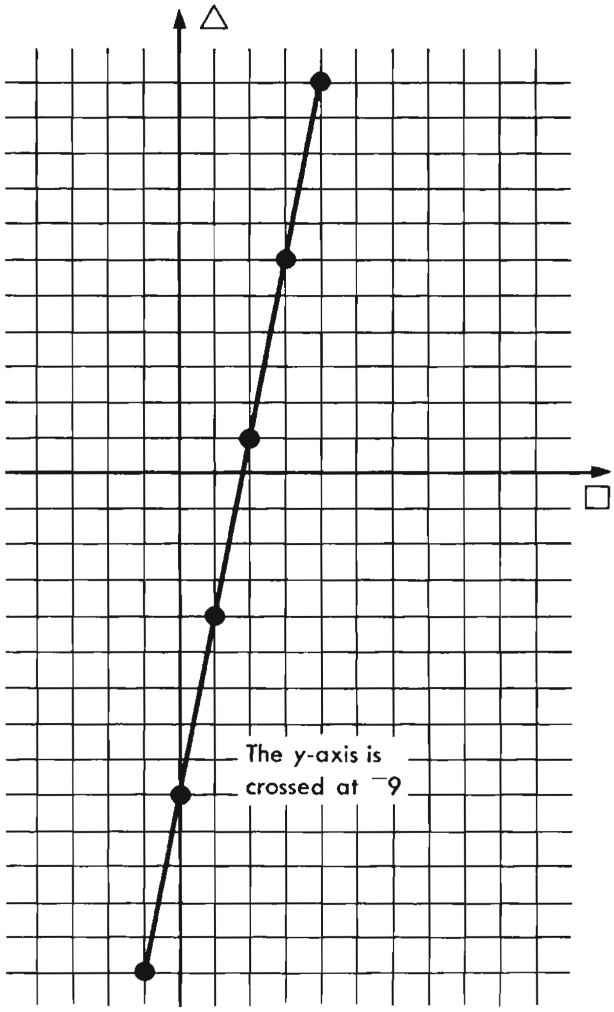
always indicates the "altitude" at which the y-axis is crossed.

For example:

$$\triangle = ({}^{-}2 \times \square) + \underset{\uparrow}{{}^{+}4}$$



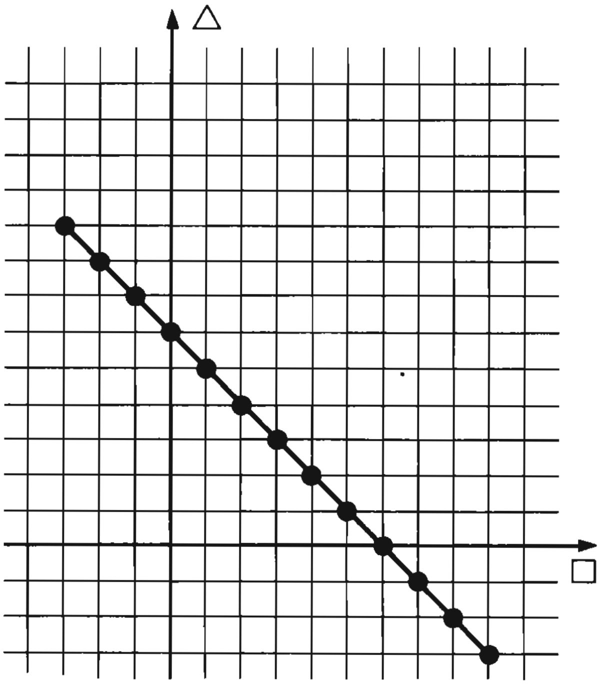
$$\triangle = ({}^{+}5 \times \square) + \underset{\uparrow}{{}^{-}9}$$



(This, of course, is the second of Myrna's two discoveries.)

(7) $\triangle = (^{-}1 \times \square) + {}^{+}6$

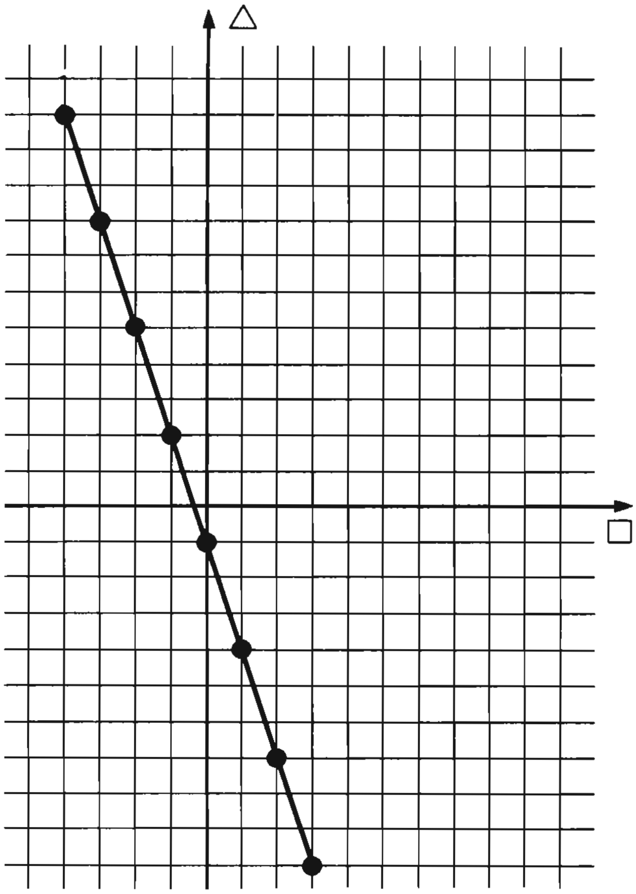
(7)



The slope pattern is: over one square to the right and down one square. The *y*-intercept is $^{+}6$ (that is, the line crosses the vertical axis at $^{+}6$).

(8) $\triangle = (^{-}3 \times \square) + {}^{-}1$

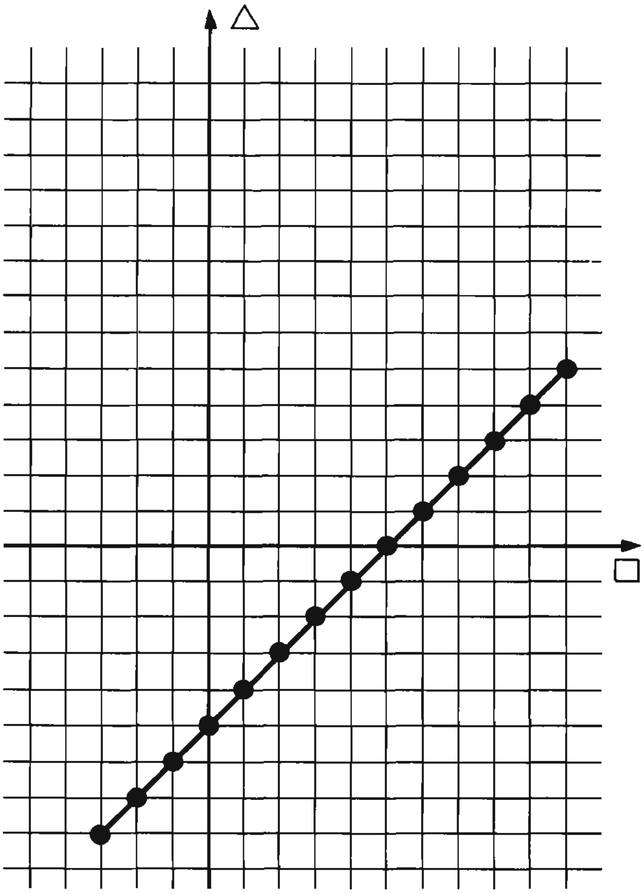
(8)



The slope pattern is: over one square to the right and down three. The *y*-intercept is $^{-}1$ (that is, the graph crosses the vertical axis at $^{-}1$).

(9) $\triangle = ({}^{+}1 \times \square) + {}^{-}5$

(9)



The slope pattern is: over one square to the right and up one square. The *y*-intercept is ⁻5 (that is, the graph crosses the vertical axis at ⁻5).

chapter 18 / *Pages 30–34 of Student Discussion Guide*

WHAT EQUATION?

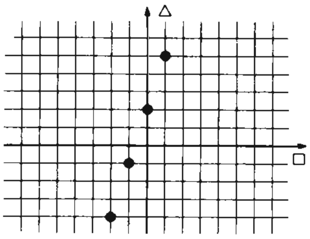
A very effective way to develop further the slope and intercept concepts is to reverse the task. Until now, the students have *started* with an equation and made a graph. In this chapter, they will start with a graph, and find the corresponding equation.

Chapter 18
WHAT EQUATION?

ANSWERS AND COMMENTS

[page 30]

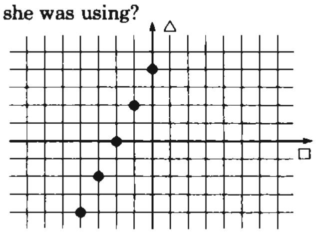
(1) Larry made this graph. What equation was he using?



$\triangle = (\_\_ \times \square) + \_\_$

(1) $\triangle = ({}^{+}3 \times \square) + {}^{+}2$

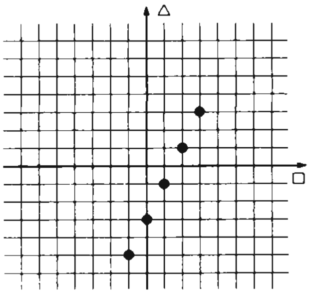
(2) Joan made this graph. Can you find the equation she was using?



$\triangle = (\_\_ \times \square) + \_\_$

(2) $\triangle = ({}^{+}2 \times \square) + {}^{+}4$

(3) Ruth made this graph. What equation did she use?

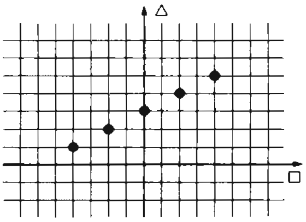


$\triangle = (\_\_ \times \square) + \_\_$

(3) $\triangle = ({}^{+}2 \times \square) + {}^{-}3$

In case you are using letters x and y, at this point, both forms of the answer are given for problems 4 through 12.

(4) Bruce made this graph. What equation do you think he was using?



$\triangle = (\_\_ \times \square) + \_\_$

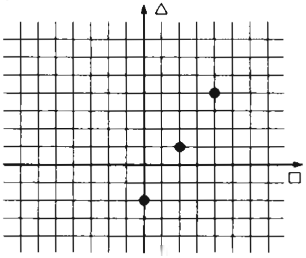
(4) $\triangle = ({}^{+}\frac{1}{2} \times \square) + {}^{+}3$

or

$y = \frac{1}{2}x + 3$

[page 31]

(5) Nancy made this graph. What equation did she use?



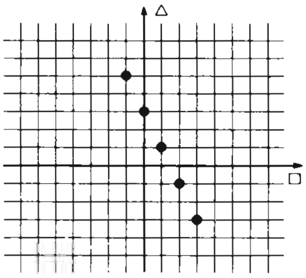
$\triangle = (\_\_ \times \square) + \_\_$

(5) $\triangle = ({}^{+}\frac{3}{2} \times \square) + {}^{-}2$

or

$y = \frac{3}{2}x + {}^{-}2$

(6) Alan made this graph. Can you find the equation he used?



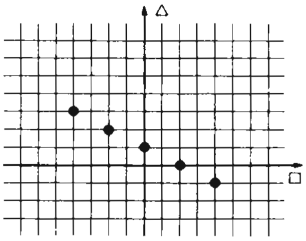
$\triangle = (\_\_ \times \square) + \_\_$

(6) $\triangle = ({}^{-}2 \times \square) + {}^{+}3$

or

$y = {}^{-}2x + {}^{+}3$

(7) Liz made this graph. What equation did she use?



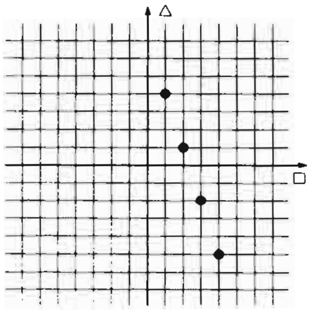
$\triangle = (\_\_ \times \square) + \_\_$

(7) $\triangle = ({}^{-}\frac{1}{2} \times \square) + {}^{+}1$

or

$y = {}^{-}\frac{1}{2}x + {}^{+}1$

(8) Kathy made this graph. What equation did she use?



$\triangle = (\_\_ \times \square) + \_\_$

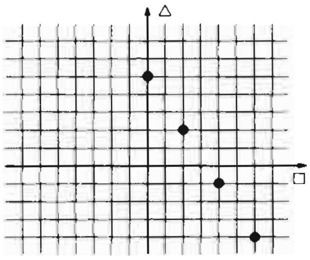
(8) $\triangle = \left({}^{-}3 \times \square\right) + {}^{+}7$

or

$y = {}^{-}3x + {}^{+}7$

[page 32]

(9) Lex made this graph. What equation did he use?



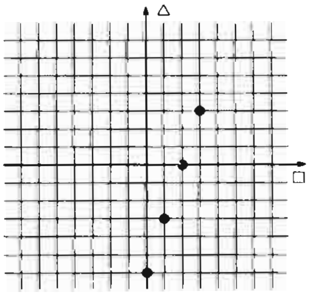
$\triangle = (\_\_ \times \square) + \_\_$

(9) $\triangle = \left({}^{-}\frac{3}{2} \times \square\right) + {}^{+}5$

or

$y = {}^{-}\frac{3}{2}x + {}^{+}5$

(10) Jill made this graph. What equation did she use?



$\triangle = (\_\_ \times \square) + \_\_$

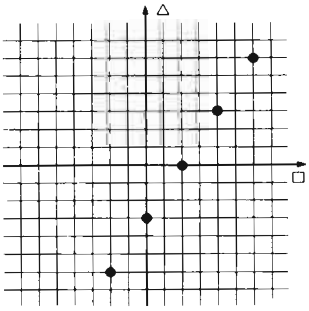
(10) $\triangle = \left({}^{+}3 \times \square\right) + {}^{-}6$

or

$y = 3x + {}^{-}6$

Can you find the equation for each of these graphs?

(11)

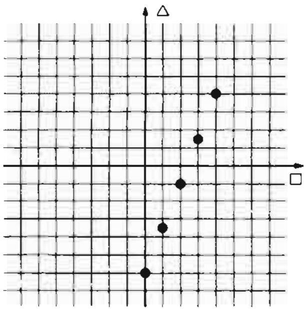


(11) $\triangle = \left({}^{+}\frac{3}{2} \times \square\right) + {}^{-}3$

or

$y = \frac{3}{2}x + {}^{-}3$

(12)



(12) $\triangle = \left({}^{+}\tfrac{5}{2} \times \square\right) + {}^{-}6$

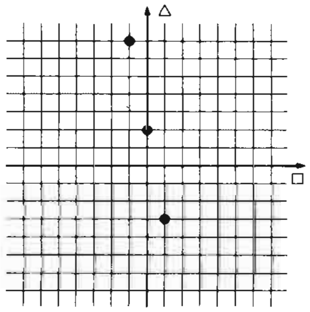
or

$y = \tfrac{5}{2}x + {}^{-}6$

[page 33]

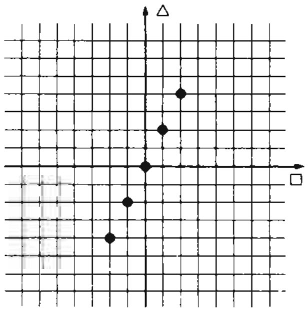
Can you find the equation for each of these graphs?

(13)



(13) $\triangle = \left({}^{-}5 \times \square\right) + {}^{+}2$

(14)

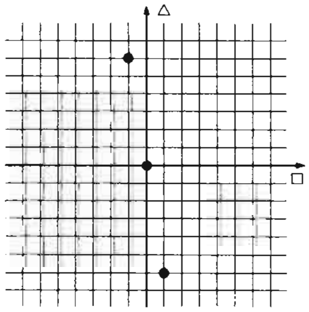


(14) $\triangle = \left(2 \times \square\right) + 0$

or, simply,

$\triangle = 2 \times \square$

(15)

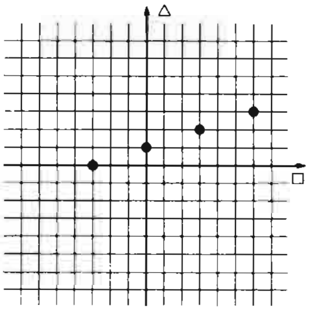


(15) $\triangle = \left({}^{-}6 \times \square\right) + 0$

or

$\triangle = {}^{-}6 \times \square$

(16)

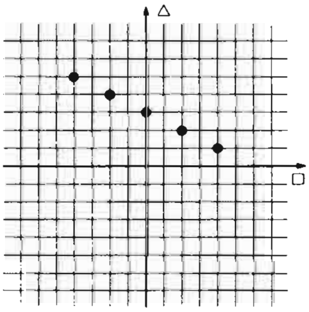


(16) $\triangle = ({}^{+}\tfrac{1}{3} \times \square) + {}^{+}1$

[page 34]

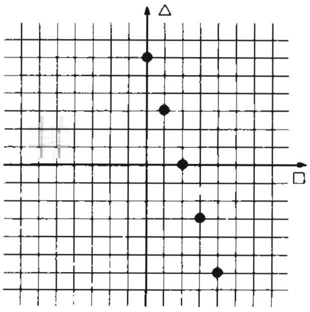
Can you find the equation for each of these graphs?

(17)



(17) $\triangle = ({}^{-}\tfrac{1}{2} \times \square) + {}^{+}3$

(18)



(18) $\triangle = ({}^{-}3 \times \square) + {}^{+}6$

chapter 19 / *Pages 34–36 of Student Discussion Guide*

Boxes on Both Sides

This chapter is intended mainly as one more link in a chain of mathematical experiences and concepts that leads eventually to:

(a) balance pictures,
(b) equivalent equations,
(c) transform operations.

The main new material of this chapter centers, as the title suggests, around examples such as:

$$(5 \times \square) + 3 = (4 \times \square) + 9.$$

The students will presumably first approach such problems by trial and error. But after some experience with problems of this sort, they will inevitably be led to the ideas of balance *pictures and transform operations.*

If the idea of *derivations* is the most important in this course and perhaps the most demanding of teacher effectiveness, the idea of *equivalent equations* and *transform operations* is probably the second most demanding upon the teacher.

Chapter 19
Boxes on Both Sides [page 34]

Can you find the truth set for each equation?

(1) $(3 \times \square) + 5 = 11$

(2) $(^+2 \times \square) + {}^-5 = {}^-15$

Answers and Comments

(1) $\{\mathbf{2}\}$

(2) $\{\mathbf{{}^-5}\}$

Note on language:

(a) This is an *open sentence*

$$(^+2 \times \square) + {}^-5 = {}^-15.$$

It is also known as an *equation.*

(b) This is the *truth set* for the open sentence above:

$$\{{}^-5\}.$$

Braces as used here always indicate a *set.*

(c) The number $^-5$ is called a *root* or *solution* of the equation $(^+2 \times \square) + {}^-5 = {}^-15$.

(d) The number $^-5$ is not the same thing as $\{{}^-5\}$, since $^-5$ is a *number,* whereas $\{{}^-5\}$ is a *set.* The *number* $^-5$ *belongs to* the *set* $\{{}^-5\}$.

This distinction becomes clearer when several numbers are involved. For example, for the equation

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

there are two *roots*, namely the two numbers 2 and 3. The number 2 is a solution of the equation. The number 3 is a solution of the equation. Therefore, the *set* of solutions of the equation is {2, 3}.

The distinction is similar to the distinction between a *person* versus a *family*. If the family consists of several people, the distinction is clear. But even if a family consists of a single person, the *person* and the *family* are conceptually distinct.

As another example, the *empty set* {▩} is a *set* with nothing in it. Nonetheless, *it* itself is a perfectly good set.

Question: How many elements are there in the set { {▩} }?

Answer: One, namely the empty set {▩}.

This distinction is covered rather nicely in various books on set theory.

Perhaps the main idea here is to remember that braces are not put around all answers. *Braces are used to indicate a set or collection, and only then.* This rule applies whether the collection consists of *several things* (for example, the set of a bigamist's wives), or *of one thing* (the set of a monogamist's wives), or *of no things* (the set of members of the United States royal family).

(3) $({}^{+}2 \times \square) + {}^{-}3 = {}^{-}23$

(3) {$^{-}$**10**}

(4) $({}^{+}2 \times \square) + {}^{-}4 = {}^{+}10$

(4) {$^{+}$**7**}

If students suggest $^{+}3$ as a root for this equation, you can respond by *asking them:*

How much is $^{+}2 \times {}^{+}3$? How much is $\left({}^{+}2 \times \boxed{{}^{+}3}\right) + {}^{-}4$?

(5) $({}^{+}2 \times \square) + {}^{+}10 = {}^{-}20$

(5) {$^{-}$**15**}

(6) $(5 \times \square) + 3 = (4 \times \square) + 9$

(6) {**6**}

(7) $(3 \times \square) + 5 = (2 \times \square) + 26$

(7) {**21**}

(8) $(3 \times \square) + 4 = (2 \times \square) + 14$

(8) {**10**}

(9) $(3 \times \square) + 9 = (2 \times \square) + 59$

(9) {**50**}

(10) $(3 \times \square) + 1 = (1 \times \square) + 21$

(10) {**10**}

This problem, of course, is intended to come as a surprise. It is from such rather carefully selected surprises that the children learn to refine and perfect their first primitive methods for solving problems.

(11) $(3 \times \square) + 5 = (1 \times \square) + 65$

(11) {**30**}

Another surprise problem, continuing the same pattern as problem 10.

[page 35]

(12) $(5 \times \square) + 11 = (2 \times \square) + 86$

(12) {**25**}

This problem follows the direction of problems 10 and 11 and extends the idea one step farther. The children may, or may not, have begun to discover the pattern by now. Of course, if they have some conceptual idea of *balancing* the two sides of an equation, that will help them greatly—*but do not give them any hints just yet!* To do so is presumptuous; it reminds them that we grownups already know all the answers, and so it robs discovery of much of its joy.

(13) $(\square \times \square) - (13 \times \square) + 22 = 0$
(13) {**11, 2**}

(14) $(\square \times \square) - (187 \times \square) + 186 = 0$
(14) {**186, 1**}

(15) $(\square \times \square) - (3 \times \square) + 2 = 0$
(15) {**2, 1**}

(16) $(\square \times \square) - (18 \times \square) + 45 = 0$
(16) {**15, 3**}

(17) $(\square \times \square) - (12 \times \square) + 35 = 0$
(17) {**5, 7**}

(18) $(\square \times \square) - (103 \times \square) + 300 = 98$
(18) {**101, 2**}

It is preferable not to give the children any hints on this problem. It is something new for them to discover!

(Actually, the new idea here is very similar to problems 7 through 12 in this chapter.)

(19) $(\square \times \square) - (16 \times \square) + 65 = 10$
(19) {**5, 11**}

(20) $(\square \times \square) - (28 \times \square) + 95 = 20$
(20) {**25, 3**}

(21) $(2 \times \square) + 3 = 8$
(21) {**2**$\frac{1}{2}$}

Most of the problems in this chapter are intended to get the children unconsciously subtracting the same thing from both sides of an equation. In problems 7 through 12 this is approached by having boxes on both sides; in problems 18 through 20 there are quadratic equations with a nonzero right-hand side, and there are also fractions making the arithmetic a bit more complicated.

But the point of it all is to lead up to equivalent equations, balance pictures, and transform operations.

(22) $(2 \times \square) + 5 = 12$
(22) {**3**$\frac{1}{2}$}

(23) $(2 \times \square) + 3 = 205$
(23) {**101**}

(24) $(2 \times \square) + 3 = 105$
(24) {**51**}

Problems 23 and 24 introduce larger numbers. If a problem is difficult enough so that just plain guessing becomes inefficient, then the children begin to want some better method for solving it. Since they *want* a better method, they are usually able to devise one.

(25) $(2 \times \square) + 3 = 204$
(25) {**100**$\frac{1}{2}$}

(26) $(2 \times \square) + 7 = 28$
(26) {**10**$\frac{1}{2}$}

(27) $(3 \times \square) + 2 = 9$
(27) {**2**$\frac{1}{3}$}

Some children will very likely suggest $2\frac{1}{2}$ right away, but after that answer is discarded, someone will come up with $2\frac{1}{3}$.

(28) $(5 \times \boxed{}) + 3 = 24$

(29) $(3 \times \boxed{}) + 10 = 15$

(30) Is 3 a root of this equation?

$$(4 \times \boxed{}) + 1 = 14$$

(31) Is 3 too small or too large to be a solution of this equation?

$$(4 \times \boxed{}) + 1 = 14$$

(32) Is 4 a root of this equation?

$$(4 \times \boxed{}) + 1 = 14$$

(33) Is 4 too large, or too small?

(34) Can you find the truth set for

$$(4 \times \boxed{}) + 1 = 14?$$

(28) $\{4\frac{1}{5}\}$

(29) $\{1\frac{2}{3}\}$

(30) **No**

(31) **Too small**

(32) **No**

(33) **Too large**

(34) $\{3\frac{1}{4}\}$

Problems 30 through 34 suggest to the children a method for solving such equations. They may have already discovered this method by themselves. The method was not presented in the earlier examples so that the children would have a fair chance to discover it themselves. Of course, they may instead have discovered the method of transform operations.

The following procedure is suggested for helping the students solve the problem:

Say:

Write on board:

too small	too large
3	4

number-line picture

too small too large

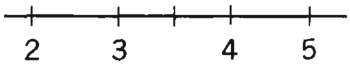
Try three and one half.

$$(4 \times \boxed{3\frac{1}{2}}) + 1 = 14$$

$$14 \quad + 1 = 14 \text{ False}$$

Three and one half is too large.

too small	too large
3	4
	$3\frac{1}{2}$



Because we have "$4 \times \boxed{}$," we can expect a denominator of four.

Let's try three and one fourth.

$$(4 \times \boxed{3\frac{1}{4}}) + 1 = 14$$

$$13 + 1 = 14 \quad \text{True}$$

The truth set is three and one fourth. $\{3\frac{1}{4}\}$

The method above, which is rather fun, could be called the "monotonicity" method. It consists of guessing, and then trying out your answer to see if it is too small, or just right, or too large. (You might call it the "method of the three bears.") This method can be used on any problem whatsoever, provided you have some method of trying out your guess and deciding whether it is too small or too large.

There is, of course, a second, entirely different method: the method of *transform operations*. This method will be developed carefully in some of the following chapters. It would be premature to tell the children about it now.

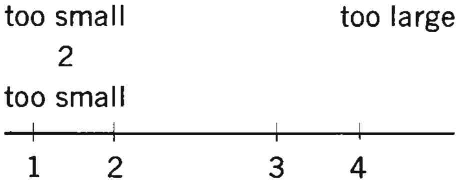
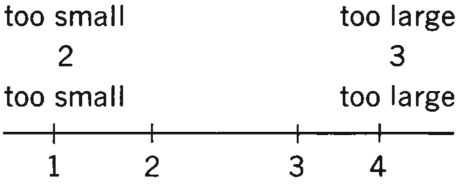
[page 36]

(35) Can you find the truth set for

$(5 \times \square) + 3 = 16$?

(35) $\{\mathbf{2\frac{3}{5}}\}$

The method here is similar to that in problems 30 through 34.

Say:	Write on board:
Suppose we try two . . .	$(5 \times \boxed{2}) + 3 = 16$ $10 + 3 = 16$ False
Two is too small . . .	
Let's try three . . .	$(5 \times \boxed{3}) + 3 = 16$ $15 + 3 = 16$ False
Three is too large.	
Because we have $5 \times \square$, we can expect a denominator of five, and we expect the answer to be one of these numbers . . .	$2\frac{1}{5}, 2\frac{2}{5}, 2\frac{3}{5}, 2\frac{4}{5}$

(Incidentally, the fancy "set" way to say this would be to say that we expect the truth set of the open sentence

$$(5 \times \square) + 3 = 16$$

to be a set containing only a single element and to be a subset of the set $\{2\frac{1}{5}, 2\frac{2}{5}, 2\frac{3}{5}, 2\frac{4}{5}\}$.)

Let's try two and three fifths . . . $(5 \times \boxed{2\frac{3}{5}}) + 3 = 16$

$13 + 3 = 16$ True

The truth set is two and three fifths. $\{2\frac{3}{5}\}$

(36) Can you find the truth set for

$(3 \times \square) + 10 = 32$?

(36) $\{\mathbf{7\frac{1}{3}}\}$

chapter 20 / *Pages 36–37 of Student Discussion Guide*

UNDOING

The point of this chapter is simple. While most third, fourth, and fifth graders will continually amaze one with their great proficiency in arithmetic, it is surprising to find that most of them will try $2\frac{1}{2}$ in an equation like $(3 \times \square) + 8 = 15$.

This shows a *good* understanding of *size* of numbers, since $2\frac{1}{2}$ is unquestionably about the right size, but it shows a surprising lack of thought about multiplicative inverses:

$$2 \times \tfrac{1}{2} = 1$$
$$3 \times \tfrac{1}{3} = 1$$
$$4 \times \tfrac{1}{4} = 1$$
$$5 \times \tfrac{1}{5} = 1$$
$$\vdots$$

If we want $3 \times \square$ to be a whole number (as it must be, if $(3 \times \square) + 8 = 15$ is to be *true*), then only whole numbers, or else fractions with denominator 3, can be substituted into the box. Instead of $2\frac{1}{2}$, we should try either $2\frac{1}{3}$ or $2\frac{2}{3}$.

To investigate this further, some bright fourth and fifth graders were asked to compute

$$\frac{21 \times 2}{2}.$$

Amazingly, they did this:

$$\frac{21 \times 2}{2} = \frac{42}{2} = 21.$$

This proves several things—you decide what.

It does, however, imply that here is something important which most children, even bright children, have somehow never happened to observe, or to reflect upon.

This is the reason for the present chapter.

ANSWERS AND COMMENTS

Chapter 20
UNDOING

[page 36]

Bart says he can find each answer by **two different** methods. Can you?

(1) $\frac{5 \times 7}{7} = ?$

(1) 5; first method—work it out in the usual rote fashion:

$$\mathbf{\frac{5 \times 7}{7} = \frac{35}{7} = 5.}$$

Second method—observe that:

(a) We start with 5.
(b) We multiply by 7.
(c) Then we divide by 7.
(d) But multiplying and dividing are inverse processes, so multiplying by 7 and then dividing by 7 gets us back where we started.
(e) Hence, the final answer is 5.

(2) $\frac{18 \times 11}{11} = ?$

(2) **18; (similar to question 1)**

(3) $\frac{21 \times 2}{2} = ?$

(3) **21**

(4) $\frac{5 \times 8}{5} = ?$

(4) **8**

(5) $\frac{1965 \times 7}{7} = ?$

(5) **1965**

(6) $\frac{31 \times 3}{3} = ?$

(6) **31**

Can you find the truth set for each equation?

(7) $(3 \times \square) + 5 = 13$

(7) $\{2\frac{2}{3}\}$

(8) $(7 \times \square) + 1 = 39$

(8) $\{5\frac{3}{7}\}$

(9) $(5 \times \square) + 10 = 61$

(9) $\{10\frac{1}{5}\}$

[page 37]

(10) $(2 \times \square) + 193 = 199$

(10) $\{3\}$

This is a change-of-pace problem, put in for the sake of variety, morale, and giving some of the other children a chance to get back into the fight.

(11) $(2 \times \square) + 5 = 12$

(11) $\{3\frac{1}{2}\}$

(12) $(2 \times \square) + 1 = 202$

(12) $\{100\frac{1}{2}\}$

(13) Jerry says that he does not know the root of this equation

$$(11 \times \square) + 5 = 40,$$

but he does know three things about it.

1. It is bigger than 3.
2. It is smaller than 4.
3. The fraction has a denominator of ______.

Do you know the denominator?

(13) **The denominator must be 11.**

Actually, we can be sure only because 11 is a prime. Otherwise there might be some cancellation, as when $3\frac{2}{4}$ reduces to $3\frac{1}{2}$ and changes denominators. The root must be an element of this set: $\{3\frac{1}{11}, 3\frac{2}{11}, 3\frac{3}{11}, 3\frac{4}{11}, 3\frac{5}{11}, 3\frac{6}{11}, 3\frac{7}{11}, 3\frac{8}{11}, 3\frac{9}{11}, 3\frac{10}{11}\}$.

Can you find the truth set for each equation?

(14) $(11 \times \square) + 5 = 40$

(14) $3\frac{2}{11}$

You use the monotonicity method, starting by splitting the interval roughly in the middle.

Write on board:

Say:

Write on board:

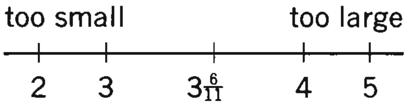
Let's try three and six elevenths . . .

$$\left(11 \times \boxed{3\tfrac{6}{11}}\right) + 5 = 40$$
$$39 \quad + 5 = 40 \quad \text{False}$$

Three and six elevenths is quite a bit too large.

(Some of the children may notice that we missed by 4* and so we can reduce $3\frac{6}{11}$ by $\frac{4}{11}$ and try $3\frac{2}{11}$.)

Let's try three and two elevenths . . .



$$\left(11 \times \boxed{3\tfrac{2}{11}}\right) + 5 = 40$$
$$35 \quad + 5 = 40 \quad \text{True}$$

This is true and so the *truth* set is three and two elevenths.

$\{3\frac{2}{11}\}$

(15)	$(7 \times \square) + 10 = 20$	(15)	**$\{1\frac{3}{7}\}$**
(16)	$(5 \times \square) + 7 = 58$	(16)	**$\{10\frac{1}{5}\}$**
(17)	$(3 \times \square) + 30 = 35$	(17)	**$\{1\frac{2}{3}\}$**
(18)	$(13 \times \square) + 10 = 144$	(18)	**$\{10\frac{4}{13}\}$**

---

* This is a pretty fancy stunt—but the children are always coming up with things like this. There doesn't seem to be any sure-fire way to hold them back.

chapter 21 / *Pages 37–38 of Student Discussion Guide*

EQUATIONS

This chapter contains five types of equations:

(a) Those in which the denominator of the root can be determined, such as

$$(7 \times \square) + 5 = 20.$$

(b) Linear equations involving signed numbers, such as

$$({}^{+}2 \times \square) + {}^{+}10 = {}^{-}30.$$

(c) Those with boxes on both sides, such as

$$(5 \times \square) + 10 = (3 \times \square) + 20.$$

(d) The usual sort of quadratics (in normal form, with positive integer roots), such as

$$(\square \times \square) - (28 \times \square) + 75 = 0.$$

(e) Quadratic equations with "a number on the end," such as

$$(\square \times \square) - (15 \times \square) + 45 = 1.$$

(f) Quadratic equations with signed numbers, such as

$$(\square \times \square) - ({}^{+}1 \times \square) + {}^{-}6 = 0.$$

The children have been told no methods for solving any of these equations, as yet. But they have been shown somewhat casually, and unobtrusively, the monotonicity method.

Chapter 21
EQUATIONS

[page 37]

Can you find the truth set for each open sentence?

(1) $({}^{+}2 \times \square) + {}^{-}10 = {}^{-}20$

(2) $({}^{+}2 \times \square) + {}^{-}100 = {}^{-}150$

(3) $({}^{+}2 \times \square) + {}^{+}10 = {}^{-}30$

(4) $({}^{+}3 \times \square) + {}^{+}5 = {}^{-}13$

(5) $({}^{+}3 \times \square) + {}^{-}7 = {}^{-}10$

(6) $({}^{+}10 \times \square) + {}^{-}11 = {}^{-}61$

ANSWERS AND COMMENTS

(1) $\{{}^{-}\mathbf{5}\}$

(2) $\{{}^{-}\mathbf{25}\}$

(3) $\{{}^{-}\mathbf{20}\}$

(4) $\{{}^{-}\mathbf{6}\}$

(5) $\{{}^{-}\mathbf{1}\}$

(6) $\{{}^{-}\mathbf{5}\}$

(7) $({}^{-}2 \times \square) + {}^{+}4 = {}^{+}10$

(7) $\{{}^{-}\mathbf{3}\}$

If the children suggest ${}^{+}3$ as the answer, rather than agreeing or disagreeing, counter with the question: How much is $({}^{-}2 \times {}^{+}3) + {}^{+}4 =$ ____?

They can surely answer this second question, and after that they can return and correctly evaluate their answer to the first question; that is, they will know that ${}^{+}3$ is wrong.

(8) $({}^{-}2 \times \square) + {}^{+}10 = {}^{+}20$

(8) $\{{}^{-}\mathbf{5}\}$

If any of the children are unsure about some of these products of signed numbers—for example, how much is ${}^{-}2 \times {}^{-}5$?—it *usually* suffices to stand by silently and let them argue among themselves. Of course, at times it is necessary to intervene.

(9) $({}^{-}2 \times \square) + {}^{+}100 = {}^{+}150$ — (9) $\{{}^{-}\mathbf{25}\}$

(10) $({}^{-}3 \times \square) + {}^{+}15 = {}^{+}3$ — (10) $\{{}^{+}\mathbf{4}\}$

(11) $(\square \times \square) - ({}^{+}5 \times \square) + {}^{+}6 = 0$ — (11) $\{{}^{+}\mathbf{2}, {}^{+}\mathbf{3}\}$

[page 38]

(12) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{+}6 = 0$ — (12) $\{{}^{-}\mathbf{2}, {}^{-}\mathbf{3}\}$

(13) $(\square \times \square) - ({}^{+}1 \times \square) + {}^{-}6 = 0$ — (13) $\{{}^{+}\mathbf{3}, {}^{-}\mathbf{2}\}$

(14) $(\square \times \square) - ({}^{+}3 \times \square) + {}^{-}10 = 0$ — (14) $\{{}^{+}\mathbf{5}, {}^{-}\mathbf{2}\}$

(15) $(\square \times \square) - ({}^{+}9 \times \square) + {}^{-}22 = 0$ — (15) $\{{}^{+}\mathbf{11}, {}^{-}\mathbf{2}\}$

(16) $(\square \times \square) - ({}^{+}7 \times \square) + {}^{+}10 = 0$ — (16) $\{{}^{+}\mathbf{5}, {}^{+}\mathbf{2}\}$

(17) $(\square \times \square) - (28 \times \square) + 75 = 0$ — (17) $\{\mathbf{25}, \mathbf{3}\}$

(18) $(\square \times \square) - (3 \times \square) + 2 = 0$ — (18) $\{\mathbf{2}, \mathbf{1}\}$

(19) $(\square \times \square) - (395 \times \square) + 394 = 0$ — (19) $\{\mathbf{394}, \mathbf{1}\}$

(20) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{-}50 = 0$ — (20) $\{{}^{-}\mathbf{10}, {}^{+}\mathbf{5}\}$

(21) $(\square \times \square) - ({}^{+}98 \times \square) + {}^{-}200 = 0$ — (21) $\{{}^{+}\mathbf{100}, {}^{-}\mathbf{2}\}$

(22) $(\square \times \square) - ({}^{-}9 \times \square) + {}^{-}22 = 0$ — (22) $\{{}^{-}\mathbf{11}, {}^{+}\mathbf{2}\}$

(23) $(5 \times \square) + 3 = (4 \times \square) + 9$ — (23) $\{\mathbf{6}\}$

(24) $(5 \times \square) + 10 = (3 \times \square) + 20$ — (24) $\{\mathbf{5}\}$

(25) $(5 \times \square) + 100 = (4 \times \square) + 107$ — (25) $\{\mathbf{7}\}$

(26) $(5 \times \square) + 1 = (4 \times \square) + 23$ — (26) $\{\mathbf{22}\}$

(27) $(5 \times \square) + 1 = (3 \times \square) + 23$ — (27) $\{\mathbf{11}\}$

(28) $(5 \times \square) + 10 = (3 \times \square) + 60$ — (28) $\{\mathbf{25}\}$

(29) $(5 \times \square) + 10 = (2 \times \square) + 70$ — (29) $\{\mathbf{20}\}$

(30)	$(2 \times \square) + 5 = 10$	(30)	$\{\mathbf{2\frac{1}{2}}\}$
(31)	$(2 \times \square) + 100 = 201$	(31)	$\{\mathbf{50\frac{1}{2}}\}$
(32)	$(7 \times \square) + 5 = 36$	(32)	$\{\mathbf{4\frac{3}{7}}\}$
(33)	$(\square \times \square) - (7 \times \square) + 12 = 2$	(33)	$\{\mathbf{2, 5}\}$
(34)	$(\square \times \square) - (15 \times \square) + 45 = 1$	(34)	$\{\mathbf{4, 11}\}$
(35)	$(\square \times \square) - (53 \times \square) + 102 = 0$	(35)	$\{\mathbf{51, 2}\}$
(36)	$(\square \times \square) - (104 \times \square) + 305 = 2$	(36)	$\{\mathbf{101, 3}\}$

chapter 22 / *Pages 39–40 of Student Discussion Guide*

TOO BIG OR TOO SMALL

One purpose of this chapter is to give the students some more experience with the number line* and with the ideas of how large numbers are.

The number line gives, among other things, a geometrical picture of the *size* of numbers, which is extremely useful (as a cognitive map in the sense of Tolman).

Chapter 22
TOO BIG OR TOO SMALL

[page 39]

(1) Can you find the truth set for this open sentence?

$$3 + \square = 5$$

(2) Without finding the answer for this equation, guess whether it will be a whole number or a fraction.

$$3 + (11 \times \square) = 28$$

(3) Can you prove it?

ANSWERS AND COMMENTS

(1) {2}

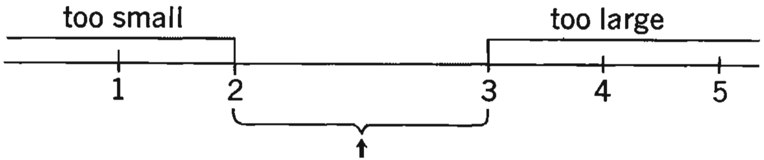
This problem, of course, is just to get all of the students thinking about open sentences.

(2) **Fraction**

(3) **It is easily proved that the answer must be a fraction (or, at least, *not* a whole number) by observing that, if a *larger* number is put into the box, then $3 + (11 \times \square)$ becomes larger, and if 2 is put into the box, then $3 + (11 \times \boxed{2})$ is 25 and is too small, whereas putting 3 into the box produces $3 + (11 \times \boxed{3})$ which is 36 and is consequently too large.**

The answer, if there is one, must certainly be greater than 2, and yet less than 3.

This can be shown on a number line as follows:



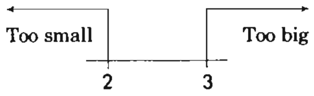
The answer, if any, must lie in this interval $\{x \mid 2 < x < 3\}$.

---

* The number line is, of course, an essential tool for nearly all of the "new mathematics curricula." In itself it is not new, having been used at least as early as in the work of Rene Descartes (1596–1650). For an interesting account of this (and many similar matters), see James R. Newman, *The World of Mathematics*, Vol. 1, p. 235 ff. (Simon and Schuster, New York, 1956).

(4) Bill uses this picture to help him solve the equation

$$3 + (11 \times \square) = 28.$$



Does this picture help?

(4) See answer to question 3.

(5) Al says that he thinks $2\frac{1}{2}$ may be the answer for the equation

$$3 + (11 \times \square) = 28.$$

Without computing, can you explain why $2\frac{1}{2}$ **cannot** be the answer to this equation?

(5) The number $2\frac{1}{2}$ cannot be an answer, because the term $(11 \times \square)$ tells us that the answer must be a fraction with denominator 11, not denominator 2.

(6) In the equation

$$3 + (11 \times \square) = 28,$$

is $(11 \times \square)$ a fraction or a whole number?

(6) Evidently, the term $(11 \times \square)$ must be a whole number, since $3 + (11 \times \square)$ must equal 28, which is a whole number.

Actually, we can compute and observe that, in fact,

$$(11 \times \square)$$

must be 25, in order to get $3 + 25 = 28$.

However, no good mathematician ever *computes* if he can answer the question by *reasoning* alone. Since we start with a whole number (3), and wish to add something $[(11 \times \square)]$ so as to get a whole number result (28) the *something* that we add *must itself be a whole number*, as we can see by this *reasoning, without* computing to find *which* whole number it is.

(7) What denominator must the answer have?

(7) It must have denominator 11.

(8) Marie says that the answer must be a fraction between 2 and 3 with a denominator of 11. Do you agree with Marie?

(8) Marie is right.

(9) Can you use Bill's picture and Marie's remark to help you find the truth set for the equation

$$3 + (11 \times \square) = 28?$$

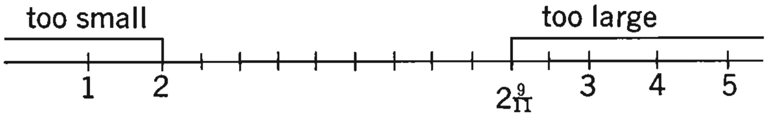
(9) It's a good picture—it should help.

(10) Jerry tried $2\frac{9}{11}$ in the equation. Was it too small or too big?

(10) Too big.

(11) After Jerry had tried $2\frac{9}{11}$, how should we draw Bill's picture?

(11) We now know that $2\frac{9}{11}$ is *too large*, so the number-line picture looks like this:



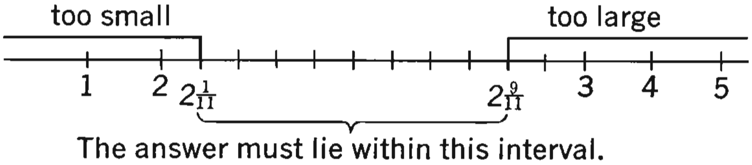
Of course, the answer must now be some element of the set $\{2\frac{1}{11}, 2\frac{2}{11}, 2\frac{3}{11}, 2\frac{4}{11}, 2\frac{5}{11}, 2\frac{6}{11}, 2\frac{7}{11}, 2\frac{8}{11}\}$.

(12) Can you find a number that is **too small**? How does this change Bill's picture?

(12) Any element of the following set would be a good answer:

$$\{2\tfrac{1}{11}, 2\tfrac{2}{11}\}.$$

Suppose they say "$2\frac{1}{11}$." The number-line picture now looks like this:



(13) Can you find a number that is **too big**? How will this change Bill's picture?

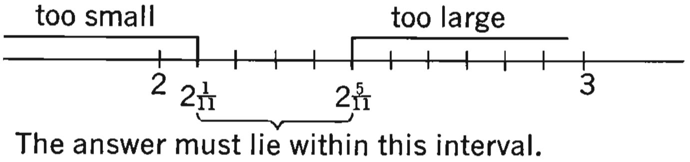
(13) **Actually, it is most efficient to try to divide the remaining interval approximately in half with each new guess. This will, in the long run, solve *most* problems with maximum speed and efficiency.**

Suppose, following this idea, we try to divide the interval

$$2\frac{1}{11} < x < 2\frac{9}{11}$$

in approximately its center. To do this, we might guess $2\frac{5}{11}$. Let's try $2\frac{5}{11}$: $3 + \left(11 \times \boxed{2\frac{5}{11}}\right)$ is, of course, 30, and is too large.

The picture now looks like this:



The number-line picture above gives an excellent geometrical representation. We could, if we wished, express this instead in the language of sets, as: The answer must be an element of the set $\{2\frac{2}{11}, 2\frac{3}{11}, 2\frac{4}{11}\}$.

[page 40]

(14) Can you find the truth set for the equation

$$3 + (11 \times \square) = 28?$$

(14) $\{2\frac{3}{11}\}$

Can you find the truth sets for these open sentences?

(15) $5 + (13 \times \square) = 29$

(15) **Proceed as in questions 2 through 14 to find the truth set $\{1\frac{11}{13}\}$.**

(16) $3 + (7 \times \square) = 12$

(16) $\{1\frac{2}{7}\}$ **(See problem 15.)**

(17) $9 + (13 \times \square) = 50$

(17) $\{3\frac{2}{13}\}$ **(See problem 15.)**

(18) $3 + (17 \times \square) = 88$

(18) $\{5\}$ **(See problem 15.)**

This problem is a typical illustration of one of our beliefs: one should not always end with *hard* problems, which may mean that the weaker students in the class always leave with a bad taste—or they may even learn that the first 15 minutes of class are for them and for the last 15 minutes of class they may as well stop trying, since the end of the lesson is always way over their heads. This is avoided by mixing hard and easy problems and by very frequently ending lessons with easy problems that *all* the children in class can do.

chapter 23 / *Pages 40–42 of Student Discussion Guide*

EQUIVALENT EQUATIONS

The official purpose of this chapter is to introduce the ideas of *equivalent equations* and *transform operations*. It has, however, an important secondary purpose: to introduce the basic pattern or idea behind the *distributive law*. The distributive law is not, as yet, identified explicitly; nor is the name *distributive law* introduced.

Chapter 23
EQUIVALENT EQUATIONS

[page 40]

(1) Jerry says that

$$(5 \times \square) + 1 = 11$$

has the **same** truth set as

$$(5 \times \square) + 2 = 12.$$

Do you agree?

(2) Alan says that

$$(3 \times \square) + 100 = 112$$

has the same truth set as

$$(3 \times \square) + 101 = 113.$$

Do you agree?

(3) Can you make up a new equation that will have the same truth set as this one?

$$(2 \times \square) + 8 = 24$$

ANSWERS AND COMMENTS

(1) **Jerry is right; both equations have the truth set $\{2\}$.**

(2) **Alan is right.**

(3) **Here are a few of the possible answers:**

$$(2 \times \square) + 9 = 25$$

$$(2 \times \square) + 10 = 26$$

$$(2 \times \square) + 108 = 124$$

$$(2 \times \square) + 18 = 34$$

$$(2 \times \square) + 7 = 23$$

$$\square + 2 = 10$$

$$\tfrac{1}{2} \times \square = 4$$

$$(4 \times \square) + 16 = 48$$

$$(1 \times \square) + 4 = 12$$

$$\square + 4 = 12$$

$$\square + 3 = 11$$

$$\square - 2 = 6$$

$$\frac{\square}{8} = 1$$

$$(\square \times \square) - (16 \times \square) + 64 = 0$$

(4) Can you make up still a **different** equation that will have the same truth set? **[page 41]**

(5) Do you know what we call two different equations that have the same truth set?

(4) See the answer to question 3.

(5) Two equations with the same *truth set* are called *equivalent equations*. This, of course, is a rhetorical question used to introduce the idea. The children are not expected to know the answer.

Examples:

These two equations are equivalent, since they have the same truth set:

$$(7 \times \square) + 5 = 26 \quad \{3\}$$

$$\square + 12 = 15 \quad \{3\}$$

These two equations are equivalent, since they have the same truth set:

$$(5 \times \square) + 10 = (3 \times \square) + 50 \quad \{20\}$$

$$\square + {}^{-}10 = {}^{+}10 \qquad \{20\}$$

These two equations are *not* equivalent, since they have different truth sets:

$$(5 \times \square) + 6 = (3 \times \square) + 7 \quad \{\tfrac{1}{2}\}$$

$$\square + 3 = 8 \qquad \{5\}$$

These two equations are equivalent, since they have the same truth set:

$$\square + 2 = 10 \quad \{8\}$$

$$(\square \times \square) - (16 \times \square) + 64 = 0 \quad \{8\}$$

These two equations are equivalent, since they have the same truth set:

$$(\square \times \square) - (5 \times \square) + 6 = 0 \quad \{2, 3\}$$

$$[(5 \times \square) - 10] \times [(2 \times \square) - 6] = 0 \quad \{2, 3\}$$

One can use this notion of *equivalence* for open sentences, even if they are not equations.

These two open sentences *are equivalent*, since they have the same truth set:

Using integers only,

$$5 < 3 + \Box < 12$$

has the truth set {3, 4, 5, 6, 7, 8}.

Using integers only,

$$4 < 2 + \Box < 11$$

has the truth set {3, 4, 5, 6, 7, 8}.

These two open sentences are equivalent, since they have the same truth set:

________________________________ was president of the United States in 1940.

{Franklin Delano Roosevelt}

________________________________ was president of the United States for longer than any other president.

{Franklin Delano Roosevelt}

These two equations are equivalent, since they have the same truth set:

$$(\Box \times \Box) - (16 \times \Box) + 55 = 0 \quad \{5, 11\}$$
$$(\Box \times \Box) - (16 \times \Box) + 85 = 30 \quad \{5, 11\}$$

These two equations are *not* equivalent, since they do *not* have the same truth set:

$$(\Box \times \Box) - (10 \times \Box) + 21 = 0 \quad \{7, 3\}$$
$$(\Box \times \Box) - (11 \times \Box) + 28 = 0 \quad \{7, 4\}$$

These two equations are *not* equivalent, since they do *not* have the same truth set:

$$(\Box \times \Box) - (8 \times \Box) + 12 = 0 \quad \{6, 2\}$$
$$(\Box \times \Box) - ({}^-8 \times \Box) + {}^+12 = 0 \quad \{{}^-6, {}^-2\}$$

These two equations are equivalent, since they *do* have the same truth set (in each case, the *empty* or *null* set):

$$\Box + 7 = \Box \quad \{▨\}$$
$$\Box - \Box = 30 \quad \{▨\}$$

These two open sentences are equivalent, since they have the same truth set:

Using integers only,

$$5 < 3 + \square < 8$$

has the truth set $\{3, 4\}$.

$$(\square \times \square) - (7 \times \square) + 12 = 0 \ \{3, 4\}.$$

These two equations are equivalent, since they have the same truth set:

$$\square + \square + 10 = 36 \ \{13\}$$

$$(2 \times \square) + 10 = 36 \ \{13\}$$

(6) Are these equations **equivalent?** $(3 \times \square) + 7 = 22$ $(3 \times \square) + 107 = 122$	(6) **Yes**
Which of these pairs are equivalent?	
(7) $(3 \times \square) + 5 = 35$ $(3 \times \square) + 105 = 135$	(7) **Equivalent**
(8) $(5 \times \square) + 2 = 22$ $(5 \times \square) + 10 = 22$	(8) **Not equivalent**
(9) $(2 \times \square) + 7 = 17$ $(2 \times \square) + 9 = 19$	(9) **Equivalent**
(10) $\square + 3 = 7$ $(2 \times \square) + 6 = 14$	(10) **Equivalent** Notice that this is an instance of the *distributive law.*
(11) $\square + 10 = 16$ $(2 \times \square) + 20 = 32$	(11) **Equivalent** This is another instance of the distributive law—but *do not* point this out to the children. If they see enough examples such as this, they should sooner or later arrive at a valid generalization by themselves.
(12) $\square + 5 = 7$ $\square + 10 = 14$	(12) **Not equivalent** This is a *negative* instance of the distributive law. It is included since both positive and negative instances are required to delineate a concept fully.
(13) $\square + 1 = 60$ $(2 \times \square) + 1 = 120$	(13) **Not equivalent** Another negative instance of the distributive law.
(14) $(5 \times \square) + 30 = 530$ $(5 \times \square) + 37 = 537$	(14) **Equivalent**

(15) $(37 \times \square) + 19 = 1960$

$(37 \times \square) + 21 = 1962$

(15) **Equivalent**

(16) $(101 \times \square) + 53 = 1290$

$(101 \times \square) + 54 = 1291$

(16) **Equivalent**

(17) $(3 \times \square) + 2 = 17$

$(3 \times \square) + 4 = 19$

(17) **Equivalent**

From all of these instances, it is hoped that the children will sooner or later make a valid generalization that adding the same number to both sides of an equation does not change the truth set.

(18) $\square + 7 = 10$

$(2 \times \square) + 14 = 20$

(18) **Equivalent**

The *distributuve law* again!

[page 42]

(19) $(3 \times \square) + 11 = 20$

$(6 \times \square) + 22 = 40$

(19) **Equivalent**

(20) $\square + 2 = 11$

$(5 \times \square) + 10 = 55$

(20) **Equivalent**

Can you fill in the missing number that will make each equation pair equivalent?

(21) $(2 \times \square) + 3 = 11$

$(2 \times \square) + 8 = \underset{\uparrow}{\underline{\quad}}$

(21) **16**

(22) $(5 \times \square) + 5 = 60$

$(5 \times \square) + 15 = \underset{\uparrow}{\underline{\quad}}$

(22) **70**

(23) $\square + 2 = 11$

$(7 \times \square) + 14 = \underset{\uparrow}{\underline{\quad}}$

(23) **77**

Another approach to the *distributive law.*

(24) $\square + 3 = 10$

$(5 \times \square) + \underset{\uparrow}{\underline{\quad}} = 50$

(24) **15**

(25) $\square + 1 = 16$

$(\underset{\uparrow}{\underline{\quad}} \times \square) + 2 = 32$

(25) **2**

(26) What can you **do** to an equation to get an **equivalent** equation?

(26) **Here are some answers you might hope to get:**

- **(a) You can add the same number to both sides of an equation.**
- **(b) You can multiply both sides of an equation by the same number.**

More imaginative children might infer, from this, that:

- **(c) You can subtract the same number from both sides of an equation.**
- **(d) You can divide both sides of an equation by two, or by three, etc.**

(27) What do we mean by saying that two equations are **equivalent?**

(27) **We mean that they have the same truth set.**

chapter 24 / *Pages 43–46 of Student Discussion Guide*

BALANCE PICTURES

Briefly and roughly, the idea of Tolman's* concept of *cognitive map* is that pseudo-geometric pictures can provide us with symbols that are very suitable for creative thinking.

How shall we have children think about equations? There is no single way that is fully adequate. We have already provided several methods:

(a) By *trial and error, or trial and error made more systematic* by considerations of *too small* and *too large*, with regard to information about possible denominators (Chapters 20 and 22), or by looking for *patterns* (as in quadratic equations).

(b) By use of *transform operations*† to replace an equation by a possibly simpler equation with the same truth set. (The children may already have arrived at some ideas on this subject by generalization from the problems of Chapter 23.)

Of course, many children, perhaps most children, will not yet have arrived at the generalization of *transform operation*. Perhaps they can be provided with a suitable mental picture, a visualization of what equations *really are*. Fortunately, an extremely suitable picture exists, and has been used in just this way by Professor W. Warwick Sawyer, Wesleyan University, Middletown, Conn.

This "visualization" is called a *balance picture*. The systematic presentation of *balance pictures* is the main objective of this chapter.

Chapter 24
BALANCE PICTURES

ANSWERS AND COMMENTS

[page 43]

(1) What do we mean by an **identity?**

(1) Teacher's definitions (see the section "Keep Your Language Clean") might be:

An identity is an open sentence that becomes true for every correctly made substitution.

An identity is an open sentence where every number works.

An identity is an open sentence where you can substitute *any* number into the box, triangle, etc., and the result will always be *true*.

---

* TOLMAN, E. C., *Behavior and Psychological Man* (Univeristy of California Press, Berkeley, Calif., 1958).

† A *transform operation* is an operation on an equation that leaves the truth set unchanged. For example, adding the same number to both sides of an equation is a *transform operation.*

An identity is an open sentence that is never false.

From the children, you may get a slightly less precise version of this idea, such as:

It's a problem where every number works.
You can put any number in the *box*.
One where every number works.

These children's definitions are not quite perfect, but they probably do show the main idea, and we, of course, want to respond encouragingly to the *correct* part of the child's answer, rather than to respond discouragingly to the weak part of the child's answer.

Which of the following are identities?

(2) $\square + 3 = 3 + \square$

(2) **An identity**

(3) $\square \times (\square + 1) = (\square \times \square) + 1$

(3) **Not an identity**

(4) $\square \times (\square + 1) = (\square \times \square) + \square$

(4) **An identity**

(5) $(\square \times \square) - (3 \times \square) + 2 = 0$

(5) **Not an identity (the truth set is $\{1, 2\}$).**

(6) $\square + 0 = \square$

(6) **An identity**

(7) $\square \times (\triangle + 5) = (\square \times \triangle) + (\square \times 5)$

(7) **An identity**

This problem is a special case of the distributive law. This problem was, of course, inserted here to *build readiness* for subsequent use of the distributive law, or, even more strongly, to help build up to the time when the children will discover the distributive law for themselves.

(8) $\square + (\triangle \times \bigtriangledown) = (\square \times \triangle) + \bigtriangledown$

(8) **Not an identity**

(9) $\square + (\triangle + 11) = (\square \times \triangle) +$

(9) **Not an identity**

(10) $\square + (\triangle \times 3) = (\square + \triangle) \times 3$

(10) **Not an identity**

(11) $\square \times (\triangle + 2) = (\square \times \triangle) + (\square \times 2)$

(11) **An identity**

(12) $\square \times (3 + 5) = (\square \times 3) + (\square \times 5)$

(12) **An identity**

(13) $\square \times (\triangle + 1962) = (\square \times \triangle) + (\square \times 1962)$

(13) **An identity**

Perhaps at this point the children are ready to generalize this pattern, and hence to "invent" the distributive law (or, if you prefer, to "discover" the distributive law).

(14) $\square + \square = 2 \times \square$

(14) **An identity**

(15) $\square + \square + \square = 3 \times \square$

(15) **An identity**

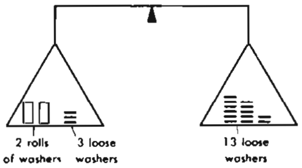
(16) $(\square + 1) \times (\square + 1) = (\square \times \square) + 2$

(16) **Not an identity**

(17) $(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$

(17) **An identity**

(18) Suppose that you have some rolls of washers (similar to the rolls of dimes or pennies you can get at the bank). You do not know how many washers there are in a roll, but you are shown a balance that just balances, like this:

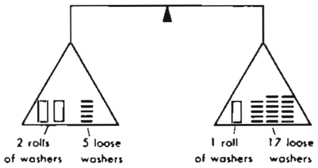


[page 44]

Can you **now** tell how many washers there are per roll? (The paper wrapping on the rolls is too light to count, so you can forget about it.)

(18) **There must be five washers per roll.**

(19) This balance just balances. How many washers are there in each roll?

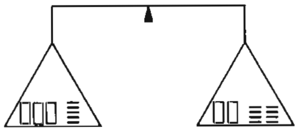


(19) **There must be 12 washers per roll.**

(20) Jerry says that for every equation like

$$(3 \times \square) + 5 = (2 \times \square) + 8$$

he can draw a "balance picture," such as:



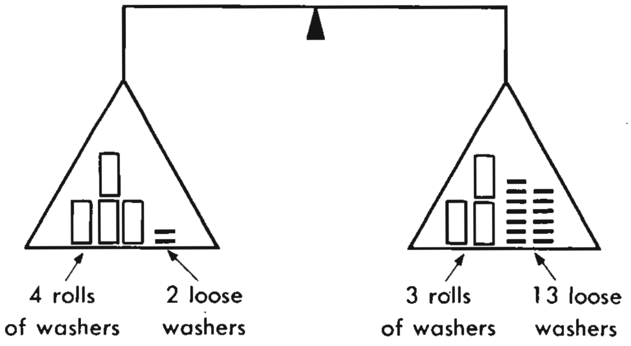
Does Jerry's picture match his equation?

(20) **Yes**

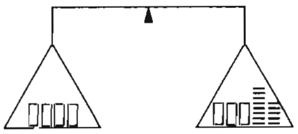
(21) Can you draw a "balance picture" for this equation?

$$(4 \times \square) + 2 = (3 \times \square) + 13$$

(21)



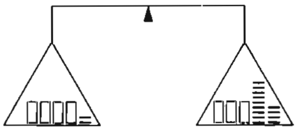
(22) Jill made this picture. Do you agree?



(22) **No; Jill left off the two loose washers that should appear on the left-hand balance pan to correspond to the 2 in the equation:**

$$(4 \times \square) + \underset{\uparrow}{2} = (3 \times \square) + 13$$

(23) Harold made this picture. Do you agree?



(23) **Yes; Harold's picture is right.**

(24) Debbie says she knows the truth set for the equation

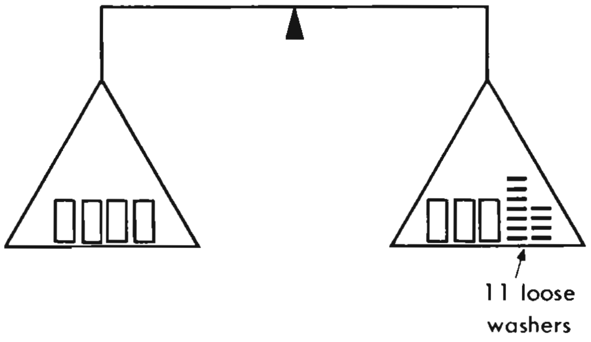
$$(4 \times \square) + 2 = (3 \times \square) + 13,$$

but she says that **if** she **didn't,** she could make the problem easier, by using Harold's balance picture.

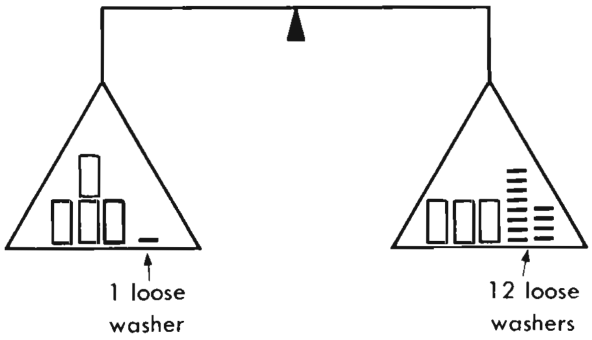
What can you do to Harold's picture to make the problem easier?

(24) Here are a few of the many possible answers:

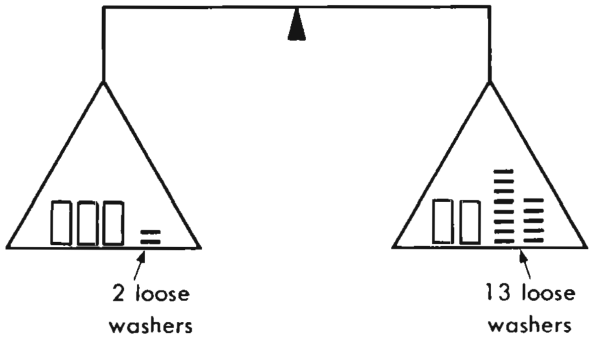
(a) You could remove two loose washers from each balance pan; the scales would still balance. The new picture would look like this:



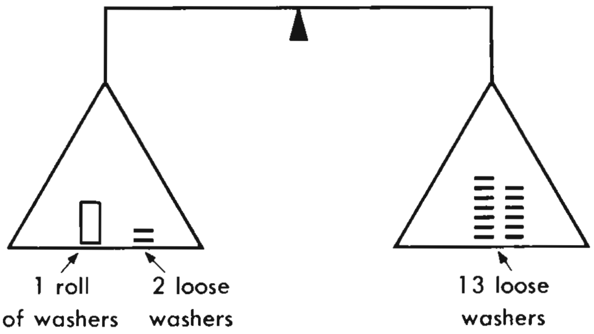
(b) You could remove one loose washer from each balance pan. The scales would still balance, and the new picture would look like this:



(c) You could remove one roll from each pan. The new picture would be:



(d) You could remove three rolls from each balance pan. The new picture would then be:



Of course, some of these choices for problem 24 are much more efficient than others. For example, (a) is reasonably efficient, in that the new picture is relatively simple; similarly, (d) is rather efficient. However, it is advisable not to push the children for efficiency just at first. Instead, let them begin with whatever choices they like, just so long as they preserve the balance by removing the same thing from both sides.

(25) What can you take off from each side of Harold's balance?

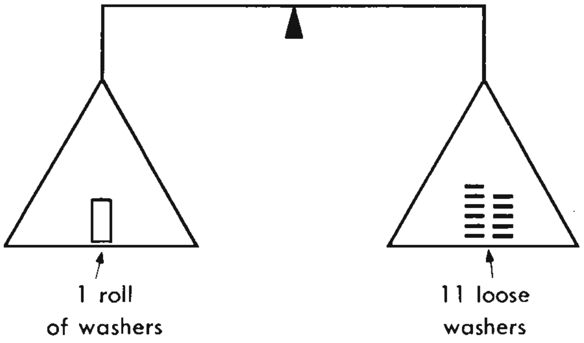
(25) See answer to question 24.

(26) What else can you take off from each side?

(26) See answer to question 24.

(27) What is the **simplest** picture you can get?

(27) Probably this is the simplest:



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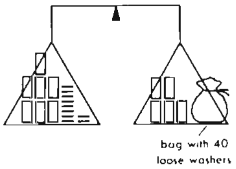
(28) Do you know the truth set for

$$(4 \times \square) + 2 = (3 \times \square) + 13?$$

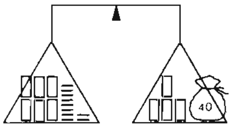
(28) {11}. This can be seen at a glance by looking at the balance picture in the answer to question 27.

(29) Try to fill in the equations to match these balance pictures.

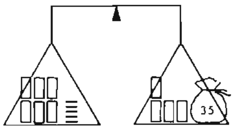
$(7 \times \square) + 10$
$= (5 \times \square) + 40$



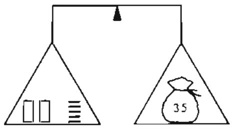
(a) ____________________



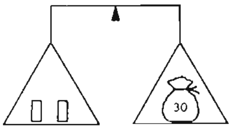
(b) ____________________



(c) ____________________



(d) ____________________



(29) (a) $(6 \times \square) + 10 = (4 \times \square) + 40$

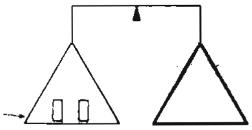
(b) $(6 \times \square) + 5 = (4 \times \square) + 35$

(c) $(2 \times \square) + 5 = 35$

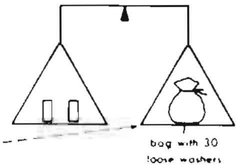
(d) $(2 \times \square) + 0 = 30$ or $(2 \times \square) = 30$

(30) Jerry says that if

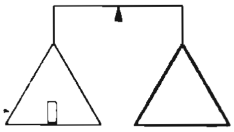
this side



balances against this side,



then **half** of this side



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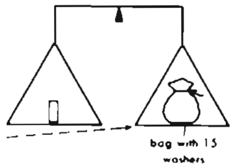
must balance against **half** of this side.



Do you agree?

(30) Yes; Jerry is right.

(31) Do you know the truth set for

$$(10 \times \square) + 7 = (8 \times \square) + 13?$$

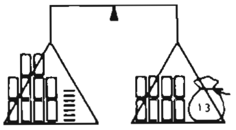
If you do, don't tell! It's a secret!

(31) {15}

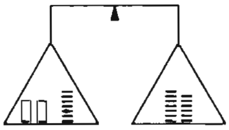
This is seen clearly from some of the preceding balance pictures, or from the simpler forms of the equation.

(32) Marie made these pictures and equations. Do you agree?

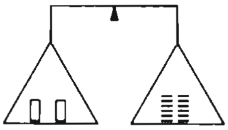
(a) $(10 \times \square) + 7 = (8 \times \square) + 13$



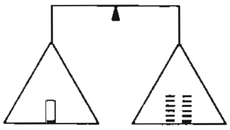
(b) $(2 \times \square) + 7 = 13$



(c) $(2 \times \square) = 12$



(d) $\square = 12$



(32) (a) Yes

(b) Yes

(c) No; Marie removed seven loose washers from the left-hand balance pan, but only one washer from the right-hand pan. The scales would not balance.

(d) No

(33) What is the truth set for

$$(10 \times \square) + 7 = (8 \times \square) + 13?$$

(33) {3}; Marie's pictures suggest the *wrong* answer, namely {12}.

chapter 25 / ***Pages 47–48 of Student Discussion Guide***

TRANSFORM OPERATIONS

This chapter works toward combining the ideas about *balance pictures* with the ideas about *equivalent equations* in order to arrive at the usual method for solving linear algebraic equations.

Chapter 25
TRANSFORM OPERATIONS

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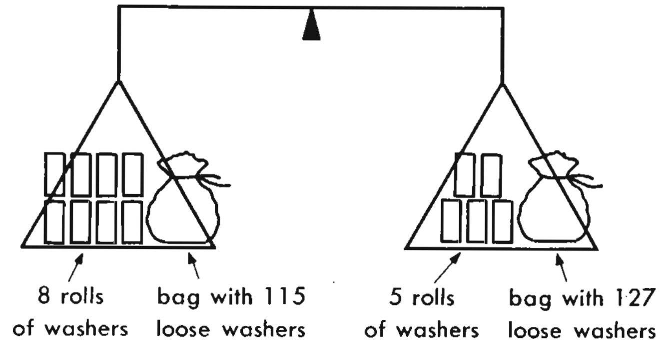
(1) Can you draw a balance picture to solve this equation?

$$(8 \times \square) + 115 = (5 \times \square) + 127$$

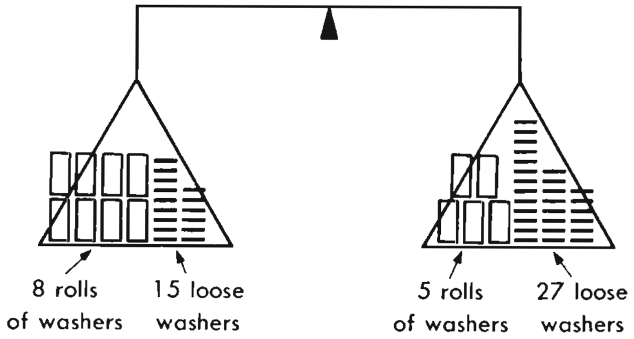
ANSWERS AND COMMENTS

(1) **One method is to proceed as follows:**



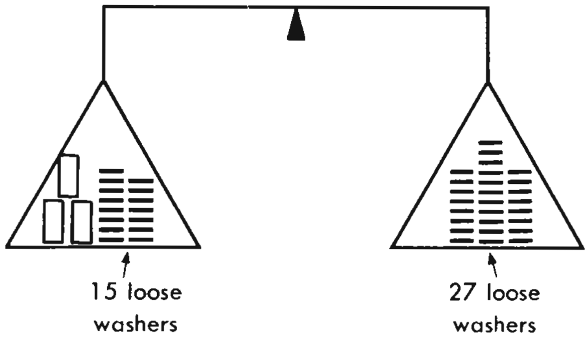
$$(\mathbf{8} \times \square) + \mathbf{115} = (\mathbf{5} \times \square) + \mathbf{127}$$

Remove 100 loose washers from each side:



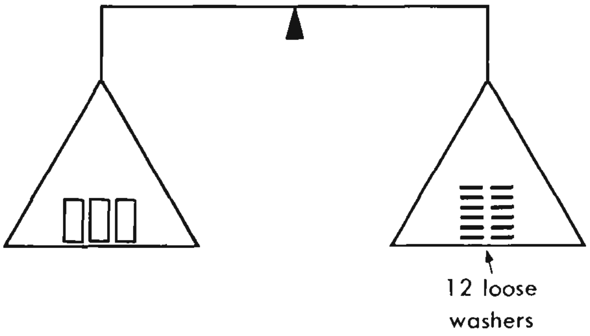
$$(\mathbf{8} \times \square) + \mathbf{15} = (\mathbf{5} \times \square) + \mathbf{27}$$

Remove five rolls from each side:



$$(\mathbf{3} \times \square) + \mathbf{15} = \mathbf{27}$$

Remove 15 loose washers from each side:



$$(3 \times \square) + 0 = 12 \quad \text{or} \quad (3 \times \square) = 12$$

By now, most children can see that the truth set must be $\{4\}$.

(2) What do we mean by **equivalent** equations?

(2) Two equations are called *equivalent* if they have the same truth set.

(3) Which of the following equations are not equivalent to

$$(8 \times \square) + 10 = 50?$$

(a) $(8 \times \square) + 11 = 51$

(b) $(8 \times \square) + 110 = 150$

(c) $(8 \times \square) + 111 = 151$

(d) $(4 \times \square) + 10 = 25$

(e) $(4 \times \square) + 5 = 25$

(f) $(8 \times \square) + 1010 = 1050$

(g) $(16 \times \square) + 20 = 100$

(h) $(16 \times \square) + 10 = 100$

(i) $(7 \times \square) + 9 = 49$

(j) $(7 \times \square) + 10 = 49$

(k) $(9 \times \square) + 10 = 50 + \square$

(l) $(10 \times \square) + 10 = 50 + (2 \times \square)$

(m) $(5 \times \square) + 5 = 25 + \square$

(n) $(5 \times \square) = 20 + \square$

(o) $(10 \times \square) = 40 + (2 \times \square)$

(3) Not equivalent: d, h, i, j.

(4) What is the **simplest** equation you can find that is equivalent to

$$(8 \times \square) + 10 = 50?$$

(4) $\square = 5$; well, let's start to simplify:

$$(8 \times \square) + 10 = 50$$

We can change this to:

$$(8 \times \square) = 40,$$

and we can change *that* to

$$\square = 5.$$

Probably most children will regard

$$(8 \times \square) = 40$$

as the best answer to this question.

(5) What is the **simplest** equation you can find that is equivalent to

$$(3 \times \square) + 7 = \square + 23?$$

(5) $\square = 8$

You may want to use this simplifying procedure (with balance pictures to illustrate each step):

$$(3 \times \square) + 7 = \square + 23.$$

Remove one roll from each pan, to get:

$$(2 \times \square) + 7 = 23.$$

Remove seven loose washers from each side, to get:

$$(2 \times \square) = 16.$$

You could stop here, or you could remove *one half* of the contents of each balance pan, to get:

$$(1 \times \square) = 8 \quad \text{or} \quad \square = 8.$$

(6) What is the simplest equation you can find that is equivalent to

$$(101 \times \square) + 2193 = (100 \times \square) + 2197?$$

(6) $\square = 4$

Proceeding as in question 5:

$$(101 \times \square) + 2193 = (100 \times \square) + 2197.$$

Remove 2000 loose washers from each balance pan:

$$(101 \times \square) + 193 = (100 \times \square) + 197.$$

(The preceding step is not one that you would probably take, but it *is* the sort of step that children often take. It is a good step, so do not worry that it may not be the best possible step.)

Remove 100 rolls from each pan, to get:

$$(1 \times \square) + 193 = (0 \times \square) + 197$$

or

$$\square + 193 = 0 + 197$$

or

$$\square + 193 = 197.$$

You could stop here, or you might remove some loose washers from each side, to get:

$$\square = 4.$$

(7) Lex says he knows five things you can do to an equation that will not change the truth set.
How many do you know?

(7) (a) Subtract the same number from each side.
(b) Add the same number to each side.
(c) Multiply both sides by the same number.
(d) Divide both sides by the same number.
(e) Use any identity.

Do not expect the children to answer this question at this stage. It is put in to get them thinking about this problem. The next question will give the children a hint, after which they probably *can* answer question 7.

Actually, of course, there is an exception to parts (c) and (d), namely that the number must not be zero, but this need not be brought up just yet.

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(8) Jackie asked Lex for a hint, and he wrote this:

(a) $3 + \square = 5$

$103 + \square = 105$

(b) $10 + \square = 25$

$5 + \square = 20$

(c) $\square + 2 = 8$

$(2 \times \square) + 4 = 16$

(d) $(8 \times \square) + 20 = 36$

$(2 \times \square) + 5 = 9$

(e) $\square + \square = 10$

$2 \times \square = 10$

Do you know Lex's "five things that won't change the truth set"?

(8) In example (a) Lex illustrates *adding the same number* to *each side*.

In example (b) Lex illustrates *subtracting the same number from each side*.

In example (c) Lex illustrates *multiplying each side by the same number* (other than zero), and incidentally also shows a correct use of the distributive law.

Example (d) illustrates *dividing each side by the same number* (other than zero).

Example (e) illustrates the use of the identity

$$\square + \square = 2 \times \square.$$

(Remember, $\square + \square = 2 \times \square$ means that $2 \times \square$ is merely another name for $\square + \square$, so all that we have done here is to take $\square + \square = 10$ and rewrite it, using a new name for $\square + \square$, to get $2 \times \square = 10$.)

(9) Do you know what we mean by a **transform operation?**

(9) A transform operation is an operation on an equation that *does not change the truth set*. The five operations listed by Lex are *transform operations*. (This is, of course, a rhetorical question for the children. Presumably they have never heard this name before and consequently could not know what it means—although they might have guessed.)

chapter 26 / *Pages 48–50 of Student Discussion Guide*

CAN YOU SOLVE THESE?

In this chapter the children practice the use of transform operations.

If, at any stage, the children misuse the distributive law, you can go back to balance pictures in order to clear things up. The minute they draw balance pictures, they will usually correct their own errors.

Chapter 26
CAN YOU SOLVE THESE?

[page 48]

ANSWERS AND COMMENTS

(1) What do we mean by a **transform operation?**

(1) **A transform operation is something that you do to an equation or inequality that does not change the truth set. (For example, adding the same number to both sides of an equation is a transform operation and multiplying both sides of an inequality by the same positive number is a transform operation.)**

(2) What transform operations can you use on

$$(8 \times \Box) + 4 = 16$$

to make it **more complicated?** To make it **simpler?**

(2) **To make the original equation more complicated:**

(a) You could add 1000 to each side:

$$(8 \times \Box) + 1004 = 1016.$$

(b) You could multiply both sides by 10:

$$(80 \times \Box) + 40 = 160.$$

(c) You could add a $\Box$ to each side:

$$(8 \times \Box) + 4 + \Box = 16 + \Box.$$

(d) You could divide both sides by 3:

$$\left(\tfrac{8}{3} \times \Box\right) + \tfrac{4}{3} = \tfrac{16}{3}.$$

(e) You could subtract a $\Box$ from each side:

$$(8 \times \Box) + 4 - \Box = 16 - \Box.$$

(f) You could use the identity

$$\frac{\Box}{1} + \Box + (0 \times \Box) + (6 \times \Box) + 0 = 8 \times \Box,$$

which says that

$$\frac{\square}{1} + \square + (0 \times \square) + (6 \times \square) + 0$$

is *another name* for $(8 \times \square)$,
to replace the simple name $(8 \times \square)$ by a more complicated name having the same meaning, to get:

$$\frac{\square}{1} + \square + (0 \times \square) + (6 \times \square) + 0 + 4 = 16.$$

There are, of course, many other ways to replace the original equation by a more complicated one. Indeed, there is no end to the possible ways of doing this.

To make the original equation simpler, you could do any of the following:

(a) Subtract 2 from each side: $(8 \times \square) + 2 = 14.$*

(b) Subtract 4 from each side: $(8 \times \square) = 12.$

(c) Divide each side by 2: $(4 \times \square) + 2 = 8.$

(d) Divide each side by 4: $(2 \times \square) + 1 = 4.$

The familiar high school method of solving equations has been defined, very appropriately, as using transform operations to make it simpler, and simpler, and simpler, until finally you get an equivalent equation that is so simple that you can see what the answer must be by merely looking.

Consider, for example, this typical solution of

$$(8 \times \square) + 4 = 16:$$

Equation	Truth Set
$(8 \times \square) + 4 = 16$	{?} (unknown)
$(2 \times \square) + 1 = 4$ (both sides divided by 4)	{?} (still unknown, but *must be the same* as in line 1)
$(2 \times \square) = 3$ (1 subtracted from both sides)	Ah—now we can see at a glance that the truth set must be $\{1\frac{1}{2}\}$. However, at each stage we have used legitimate transform operations, so we have not changed the truth set.

---

* You could, of course, argue as to whether this has really made the equation simpler, or left it about the same, or perhaps even made it more complicated. Simplicity is, after all, basically a matter of *taste*. There are no hard-and-fast rules about it.

Equation Truth Set

Hence, $\{1\frac{1}{2}\}$ must also be the truth set for

$$(2 \times \square) + 1 = 4,$$

and also for

$$(8 \times \square) + 4 = 16.$$

This latter is what we wanted to find.

In summary, the open sentence $(8 \times \square) + 4 = 16$ must have the truth set $\{1\frac{1}{2}\}$.

(3) What do we mean by **equivalent equations?**

(3) Two equations are called equivalent if they have the same truth set.

Can you find the truth sets for these open sentences?

(4) $(\square \times \square) - (13 \times \square) + 46 = 10$

(4) Subtract 10 from each side, to get

$$(\square \times \square) - (13 \times \square) + 36 = 0.$$

The two secrets now show us that the truth set must be $\{9, 4\}$.

It seems advisable not to tell the children how to solve this equation. Methods are closely kept personal secrets.

The teacher writes an equation on the board, and the children say what the truth set is. A child who has, privately, discovered a good method will usually be able to state the correct truth set. But neither teacher nor students usually mention methods. The methods are secret.

Of course, like nearly all rules, this one has its exceptions.

(5) $(\square \times \square) - (15 \times \square) + 58 = 2$

(5) **$\{8, 7\}$**

(6) $(\square \times \square) - (18 \times \square) + 70 = 25$

(6) **$\{3, 15\}$**

(7) $(1000 \times \square) + 2917 = 7917$

(7) **$\{5\}$**

(8) $(\square \times \square) - (104 \times \square) + 505 = 202$

(8) **$\{101, 3\}$**

(9) $(3 \times \square) + 3 = 25$

(9) **$\{7\frac{1}{3}\}$**

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(10) $(100 \times \square) + {}^{-}35 = {}^{-}45$

(10) **$\{{}^{-}\frac{1}{10}\}$**

(11) $({}^{+}2 \times \square) = {}^{-}5$

(11) **$\{{}^{-}2\frac{1}{2}\}$**

(12) $({}^{-}3 \times \square) = {}^{+}6$	(12) $\{{}^{-}2\}$
(13) $(\square \times \square) - (15 \times \square) + 44 = 0$	(13) $\{4, 11\}$
(14) $(\square \times \square) - (17 \times \square) = {}^{-}70$	(14) $\{10, 7\}$
(15) $\square + {}^{-}30 = {}^{+}50$	(15) $\{{}^{+}80\}$
(16) $({}^{+}3 \times \square) = {}^{-}12$	(16) $\{{}^{-}4\}$
(17) $({}^{-}5 \times \square) = {}^{+}5$	(17) $\{{}^{-}1\}$
(18) $37 + \square = 37$	(18) $\{0\}$
(19) $(\square \times \square) - ({}^{-}7 \times \square) + {}^{+}10 = 0$	(19) $\{{}^{-}5, {}^{-}2\}$
(20) $(\square \times \square) - ({}^{+}1 \times \square) + {}^{-}12 = 0$	(20) $\{{}^{+}4, {}^{-}3\}$
(21) $(\square \times \square) - ({}^{+}3 \times \square) = {}^{+}10$	(21) $\{{}^{+}5, {}^{-}2\}$
(22) $(2 \times \square) + {}^{+}8 = 0$	(22) $\{{}^{-}4\}$
(23) $(13 \times \square) = 21$	(23) $\{\frac{21}{13}\}$ or $\{1\frac{8}{13}\}$
(24) $(217 \times \square) + 100 = 1402$	(24) $\{6\}$
(25) $(5 \times \square) + 3 = 20$	(25) $\{3\frac{2}{5}\}$
(26) $(7 \times \square) + 3 = (5 \times \square) + 43$	(26) $\{20\}$
(27) $(7 \times \square) + 5 = (4 \times \square) + 38$	(27) $\{11\}$
(28) $(100 \times \square) + 21 = (99 \times \square) + 23$	(28) $\{2\}$
(29) $(\square \times \square) - (0 \times \square) + {}^{-}25 = 0$	(29) $\{{}^{+}5, {}^{-}5\}$
(30) $(\square \times \square) = {}^{+}49$	(30) $\{{}^{+}7, {}^{-}7\}$
(31) $(\square \times \square) = {}^{-}36$	(31) {▨}

That is, the solution set for problem 31 is the null set; there are no *real* numbers that will make this true. (The null set, of course, is usually denoted by the Greek letter ϕ, but we have chosen to use the symbol

{▨}

in the belief that it may be more suggestive.)

(32)	$(351 \times \square) + 2 = 2459$		(32)	$\{7\}$
(33)	$(702 \times \square) + 4 = 2110$		(33)	$\{3\}$
(34)	$\square \times \square = {}^{+}121$		(34)	$\{{}^{+}11, {}^{-}11\}$
(35)	$\square \times \square = 9$		(35)	{▨}
		[page 50]		
(36)	$\square = \frac{{}^{+}16}{\square}$		(36)	$\{{}^{+}4, {}^{-}4\}$
(37)	$(8 \times \square) + 31 = 31$		(37)	$\{0\}$
(38)	$({}^{+}2 \times \square) + {}^{+}10 = {}^{-}4$		(38)	$\{{}^{-}3\}$
(39)	$({}^{+}2 \times \square) + {}^{-}15 = 20$		(39)	$\{{}^{-}2\frac{1}{2}\}$
(40)	$({}^{-}2 \times \square) + 15 = {}^{-}10$		(40)	$\{{}^{+}2\frac{1}{2}\}$
(41)	$({}^{-}3 \times \square) + 21 = 0$		(41)	$\{{}^{+}7\}$
(42)	$({}^{-}5 \times \square) + {}^{-}15 = 0$		(42)	$\{{}^{-}3\}$

chapter 27 / *Pages 50–51 of Student Discussion Guide*

"ADDING" STATEMENTS

This chapter is a prime example of careful readiness building.

Some of the following chapters will make use of the idea of implication. Traditional curricula, when they need implication, invoke it without comment. However, children (even of high school or college age) may not be familiar with the basic concept of implication; hence, they cannot use it effectively as a tool.

In this chapter we shall try to develop something of the idea of *implication*.

Indeed, children usually are not ready for *this* chapter; hence, they need an introduction to the introduction, as it were. This is provided rather easily by tentatively bringing up this chapter in many consecutive lessons, until it finally takes hold. This is a real art and calls for a very delicate touch. One needs to avoid *premature pushing*, and at the same time, one must try to avoid the ennui of excessive repetition.

The purpose of this chapter is mathematically explained by saying that any mathematical system consists of a set of *axioms*, plus all of the statements (known as *theorems*) that are implied by these axioms.

The Madison Project material approaches this axiom-implication-theorem idea as follows:

First, the children build a tentative list of identities by the only logic they know (that is to say, by the only logic they know at the start of the course): namely, by optimistic extrapolation from a few instances. (They have some awareness that this kind of logic is not without its dangers.)

Second, two things are accomplished simultaneously:

(a) The ideas of implication and generalization are developed by using verbal examples (as in the present chapter).

(b) The children are asked if they can devise any methods for shortening the list of tentatively accepted identities.

Third—and last—*generalization* and *implication* are used to shorten the list of identities. Those identities which cannot be eliminated from the list in this way are given the name *axioms*. Those which can be eliminated, are given the name *theorems*. The fully developed chain of implications that suffices to eliminate a theorem is known as a *derivation*.

Now to return to the second item listed here. The basic approach to the important task of *list-shortening* is the same whether the list consists of verbal statements or of mathematical ones.

I. *Generalization*. As usual, this problem is given to the students: "How can you shorten lists, without really losing anything?" It is the students' job—not the teachers'—to find answers to this question.*

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* Among Project tape recordings, several bear on this point. You might be especially interested in tape number D-1, issued in 1960 and still available.

A fourth-grade girl made up this example:

"Suppose I write on a postcard,

> 'We are all fine here. I am fine. Tom is fine.
> Jeff is fine. Andrea is fine.'

Then I could cross off all of the statements except the first, without really losing anything."

That girl just discovered generalization, which is a very fine answer to the question of how we can shorten lists.

Now try her method on an algebraic example by using the following list:

$$\begin{aligned} 3 \times \square &= \square \times 3 \\ 4 \times \square &= \square \times 4 \\ 5 \times \square &= \square \times 5 \\ 1066 \times \square &= \square \times 1066 \\ 1732 \times \square &= \square \times 1732 \\ 1984 \times \square &= \square \times 1984 \\ 1{,}000{,}000 \times \square &= \square \times 1{,}000{,}000 \end{aligned}$$

This *entire* list can be replaced by the single identity:

$$\triangle \times \square = \square \times \triangle.$$

This is one of two fundamental methods for *shortening lists.* It is known as the method of *generalization.*

II. *Implication.* The following example was made up by a sixth-grade boy.

"Suppose I tell someone,

> 'My cousin plays in the Little League.
> Only boys play in the Little League.
> My cousin is a boy.'

Then I could leave off the last statement, because after I've told them, 'My cousin plays in the Little League' and 'Only boys play in the Little League,' they would then *know* that my cousin is a boy."

That sixth-grade boy just discovered implication.

Now, use the following list to try on an algebraic example.

$$\begin{gathered} \square \times \triangle = \triangle \times \square \\ \square + \triangle = \triangle + \square \\ A + (B \times C) = (C \times B) + A \end{gathered}$$

This list can be shortened by crossing off the last statement, since it is *implied* by the first two.

To show this in detail, a *derivation** is made as follows:

Theorem: $A + (B \times C) = (C \times B) + A$

Proof: $A + (B \times C) = A + (B \times C)$ A trivial identity

Commutative law for addition

$A + (B \times C) = (B \times C) + A$

Commutative law for multiplication

$A + (B \times C) = (C \times B) + A$

Q.E.D.

The preceding discussion has given a brief preview of the Madison Project work on axioms, theorems, implication, derivations, list shortening, etc.

Let us now return to the present chapter.

Chapter 27
"Adding" Statements

Answers and Comments

[page 50]	
(1) Jerry says that these two statements together sort of "add up" to a third statement. What do you think it is? (a) I have two United States coins with a total value of 15 cents. (b) The first coin is a dime. (c) ______________	(1) **(c) The second coin is a nickel.**
(2) What do these two statements "add up" to? (a) I live on the shortest street in Syracuse. (b) The shortest street in Syracuse is Maple Street. (c) ______________	(2) **(c) I live on Maple Street.**
(3) What is the third statement that these two statements add up to? (a) Al is the tallest boy in his class. (b) Al and Jim are in the same class. (c) ______________	(3) **(c) Al is taller than Jim.**
[page 51]	
(4) What is the third statement that these two statements imply? (a) Only girls belong to the sewing club. (b) Jerry's cousin belongs to the sewing club.	(4) **(c) Jerry's cousin is a girl.**
(5) Can you make up two statements that will imply a third?	(5) **There are, of course, many possibilities.**

---

* Please don't worry if this "derivation" doesn't make too much sense just yet. We shall get to this in a later chapter, by which time it should seem perfectly natural and reasonable.

Do not be disappointed if the class does not pick up this idea right away. Some classes get it right off, and enjoy it. Others need several brief, casual introductions in successive lessons before they show signs of interest or comprehension.

(6) Do these two statements imply a third?

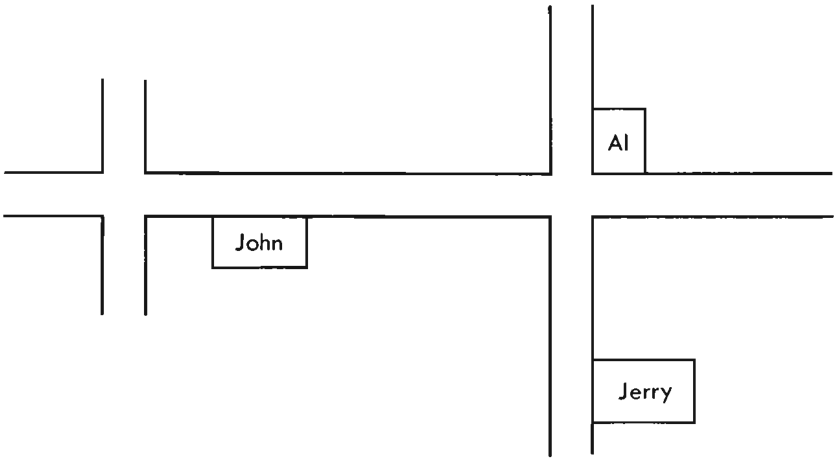
(a) John lives on the same street as Al.

(b) Al lives on the same street as Jerry.

(c) ________________

(6) **(c) John lives on the same street as Jerry.**

This is argumentative, because some of your children may point out this kind of situation:



Of course, you can define "lives on a street" to mean that their mailing address is on that street, and you can assume that each house can have only one mailing address. In that case, the two statements imply the third statement.

(7) Do these two statements imply a third?

(a) I have two coins in my hand.

(b) Their total value is 11 cents.

(c) ________________

(7) **(c) One is a dime, and the other is a penny.**

(8) This statement implies at least three others. Can you find them?

(a) One half of the people in my class are boys.

(b) ________________

(c) ________________

(d) ________________

(8) **(b) There are an even number of people in my class.**
(c) One half of the people in my class are girls.
(d) In my class, there are just as many boys as there are girls.
(e) There are at least two people in my class.
(f) There is at least one boy in my class.
(g) There is at least one girl in my class.

(9) Can you make up two statements that will imply a third?

(a) ________________

(b) ________________

(c) ________________

(9) **This is a repeat of question 5.**

Perhaps by now the response to this problem will reflect more interest and comprehension, but do not be disappointed if it does not. This lesson sometimes takes quite a while and many gentle repetitions (for a few minutes each time) to be comprehended.

Even if the idea is never fully understood, do not worry about it. These children have years of learning still ahead of them. All you need do is give them the best possible start, and this is more a matter of getting them to *like* the subject than it is of covering the book or achieving some arbitrary level of performance.

chapter 28 / *Pages 52–55 of Student Discussion Guide*

SHORTENING LISTS

We continue the strategy outlined in Chapter 27—to work through the ideas of:

(a) *Identity.* An open sentence that becomes *true* for every correctly made substitution.

(b) *Classification of open sentences*: as "not an identity" or as a "tentative identity." We *cannot* try *all* the numbers when testing for an identity; hence we cannot be certain that an open sentence is an identity. However, one false substitution will classify an open sentence definitely as *not an identity.*

But if all our substitutions produce true statements, then we cannot reject the supposed identity (since we have found no false results), and, equally, we cannot definitely classify it as an identity (since we have not tried *all* possible numbers). In this case, the final decision is up in the air, and we add the open sentence to our *tentative list of identities.*

(c) *Tentative list of identities.* This list is obtained as mentioned above. It can be made exceedingly long and varied. (There is a reason for doing so, since we want the children to become impatient with the excessively *long* list, and to try to shorten it.)

(d) *List shortening.* We can shorten a list *without losing anything,* by either of two methods:

Generalization:

Example 1

this list	*can be replaced by this list*
$5 + \square = \square + 5$	$\triangle + \square = \square + \triangle$
$6 + \square = \square + 6$	
$7 + \square = \square + 7$	
$1066 + \square = \square + 1066$	

Example 2

this list	*can be replaced by this*
Joe is fine.	We are all fine.
Mary is fine.	
Abe is fine.	
⋮	

Implication (*or inference*):

Example 1

this list	*can be replaced by this list*
My cousin plays in the Little League.	My cousin plays in the Little League.

Only boys play in the Little League.	Only boys play in the Little League.
My cousin is a boy.	

Example 2

this list	*can be replaced by this list*
$\square \times \triangle = \triangle \times \square$	$\square \times \triangle = \triangle \times \square$
$\square + \triangle = \triangle + \square$	$\square + \triangle = \triangle + \square$
$a + (b \times c) = (c \times b) + a$	

(e) *Axioms* and *theorems*. When we take the statements of algebra, and shorten the list as much as we can, those statements that remain on the minimal list are called *axioms;* those statements that can be eliminated during the list shortening are called *theorems.*

(f) *Derivations.* The precise process by which we eliminate a *theorem* is called a *derivation.*

The students look at derivations beginning with problem 33 of this chapter. Derivations form a central part of the study of modern algebra and the students will continue to work with them for many succeeding lessons.

This is a new approach to algebra. It is true to the spirit of modern mathematics, the children take to it very naturally, and—in fact—it can be *fun!*

Chapter 28
SHORTENING LISTS

ANSWERS AND COMMENTS

[page 52]

(1) What is an **identity**?

(1) This question has been asked many times before. The children are expected to answer *in their own words*—which usually means, *a fortiori*, not quite right but nonetheless showing real understanding. They may say "a problem where *any* number works"—or something like that. The careful definition (one that we want the children to *hear*, but we do not expect them to *repeat*) is: an *identity* is an open sentence that becomes *true* for every correctly made substitution.*

Which of these are identities?

(2) $3 \times \square = \square \times 3$

(2) This is an identity.

The child probably has no logic that will *prove* this, but he is probably easily convinced that it seems like a good, safe bet.

(3) $131 \times \square = \square \times 131$

(3) This is an identity.

(4) $\square \times 0 = \square$

(4) This is *not* an identity. For example, if 2 is substituted into the box, a false statement is obtained.

---

* Actually, you can even quarrel with this definition. To quote Morris Kline: "Sufficient unto the day is the rigor thereof." Let's not try for perfection—it has a way of remaining just outside our grasp.

(5) $\square \times 1 = 1$

(5) This is *not* an identity. For example, substitute 2 into the box.

(6) $\square + 0 = \square$

(6) This is an identity.

(7) $\square + (2 \times \triangle) = \square + (2 \times \triangle)$

(7) This is an identity. It is a particularly safe bet, since the equation is exactly the same on both sides.

An identity that is *exactly* the same on both sides of the equals sign is called a *trivial* identity. We can be confident that it really is an identity.

(8) $\square \times 0 = 0$

(8) An identity.

(9) $\square \times 2 = 2$

(9) Not an identity.

(10) $(\square \times \square) - (5 \times \square) + 6 = 0$

(10) Not an identity. Indeed, the truth set for this open sentence is $\{2, 3\}$; for all *other* numbers, it becomes *false*.

(11) $3 + \square = 5$

(11) Not an identity; becomes *true* only if 2 is substituted into the box; for all *other* substitutions, it becomes false (this is equivalent to saying that the truth set is $\{2\}$).

(12) $\frac{\square}{\square} = 1$

(12) An identity.

This answer provides grounds for argument because of the dubious case of:

$$\frac{0}{0}.$$

It is preferable to avoid such complications at first, *unless the children themselves suggest them.* (Hope that they don't; they're not quite ready yet.)

(13) $\square \times 1 = \square$

(13) An identity.

(14) $\square \times 2 = \square$

(14) Not an identity.

(15) $\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$

(15) Surprise! This actually *is* an identity.

(16) $(\square + 3) \times (\square + 3)$
$= (\square \times \square) + (6 \times \square) + 9$

(16) An identity.

(17) $\triangle \times \square = \square \times \triangle$

(17) An identity.

(18) $\square + \triangle = \triangle + \square$

(18) An identity.

(19) $\square = \square$

(19) An identity (*trivial*).

(20) $A + B = B + A$

(20) An identity (of course, the same as question 18 above).

(21) $A + (B \times C) = (C \times B) + A$

(21) An identity.

(22) $\square + (\triangle \times \bigtriangledown) = (\bigtriangledown \times \triangle) + \square$

(22) An identity (the same as question 21).

(23) $3 \times A = A \times 3$

(23) An identity.

(24) $B \times 0 = 0$

(24) An identity (the same as question 8).

(25) Can you shorten this list? [page 53]

(a) My cousin plays in the Little League.
(b) Only boys play in the Little League.
(c) My cousin is a boy.

(25) Delete the third statement and nothing is lost. That is to say, *if you told somebody,*

My cousin plays in the Little League.
Only boys play in the Little League.

you would have told him the same thing as if you had said,

My cousin plays in the Little League.
Only boys play in the Little League.
My cousin is a boy.

(26) Barth says that you can shorten the list in question 25. He says that if you tell somebody:

"My cousin plays in the Little League,"
and if you tell them:
"Only boys play in the Little League,"
then you don't **need** to tell them:
"My cousin is a boy."

Do you agree?

(26) Barth is right.

(27) What do we mean by **implication?** Can you give an example?

(27) Questions 25 and 26 are precisely what we mean by *implication.* Statements (a) and (b) *imply* statement (c).

(28) Brian says that he can leave out **any one** sentence on this list without losing anything.

(a) July 20 is my birthday.
(b) Today is July 20.
(c) Today is my birthday.

Do you agree?

(28) Brian is right.

(29) Suppose you wrote on a postcard:

"We are all fine here. I am fine."

Would you really need both statements?

(29) No. You could delete the second statement.

(30) Can you shorten this list?

(a) All my friends go to the Franklin School.
(b) Joe is my friend.
(c) Joe goes to the Franklin School.

(30) Delete statement (c), since (a) and (b) *imply* (c).

(31) Can you shorten this list?

(a) Half the people in my class are boys.
(b) Half the people in my class are girls.
(c) There are just as many boys in my class as there are girls.
(d) The number of people in my class is an even number.
(e) There is at least one boy in my class.
(f) There is at least one girl in my class.

(31) Keep statement (a) and delete all of the others, since (a) implies all of the others.

(32) Can you shorten this list?

(a) $3 \times \square = \square \times 3$
(b) $4 \times \square = \square \times 4$
(c) $10 \times \square = \square \times 10$

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(32) Replace *all* of these by:

$$\triangle \times \square = \square \times \triangle.$$

(d) $1066 \times \square = \square \times 1066$

(e) $1941 \times \square = \square \times 1941$

(f) $^-2 \times \square = \square \times {}^-2$

(33) Can you shorten this list?

(a) $\square + \triangle = \triangle + \square$

(b) $\square \times \triangle = \triangle \times \square$

(c) $A + (B \times C) = (C \times B) + A$

(33) Delete (c) since (a) and (b) imply it.

(34) Jerry says that he can shorten the list in question 33 by omitting the last statement. He says that the first two statements imply the third.

Do you agree?

(34) Jerry is right.

(35) Nancy says that the first statement says you can **add** in either order,

$$A + (B \times C) = (B \times C) + A,$$

and the second statement says you can **multiply** in either order,

$$A + (B \times C) = (C \times B) + A,$$

so the two together imply the third.

What do you think?

(35) Nancy is right.

If you think about questions 34 through 44, you will find that they do in fact "make sense." This is a bit hard to explain, and it is probably something one can best dig out for one's self.

In order to show that two statements imply a third, mathematicians usually make a **derivation.**

(36) Ann made this derivation:

Theorem: $A + (B \times C) = (C \times B) + A$

Proof: $A + (B \times C) = A + (B \times C)$
$A + (B \times C) = (B \times C) + A$
$A + (B \times C) = (C \times B) + A$
Q.E.D.

What do you think of Ann's derivation?

(36) Ann's derivation is correct.

(37) Why did Ann start with

$$A + (B \times C) = A + (B \times C)?$$

(37) Because it is exactly the same on both sides, and is therefore an identity in which we can place great confidence.

(38) What do we mean by a **trivial** identity?

(38) A trivial identity is one that is exactly the same on both sides.

(39) What **special** property does a trivial identity have?

(39) It is guaranteed to be an identity.

(40) Joe says Ann got from the line

$$A + (B \times C) = A + (B \times C)$$

to the line

$$A + (B \times C) = (B \times C) + A$$

by using the **commutative law for multiplication.**

Do you agree?

(40) Joe is wrong (see question 41).

[page 55]

(41) Al says no, Ann reversed the terms on the **addition sign**

$$A \underset{\uparrow}{+} (B \times C),$$

and so she must have used the **commutative law for addition.**

What do you think?

(41) Al is right.

(42) Vivian says you can take the commutative law for addition

$$\square + \triangle = \triangle + \square$$

and put A into the $\square$

$$\boxed{A} + \triangle = \triangle + \boxed{A}$$

and $(B \times C)$ into the $\triangle$

$$\boxed{A} + \overset{\triangle}{(B\times C)} = \overset{\triangle}{(B\times C)} + \boxed{A},$$

which says (in other words)

$$A + (B \times C) = (B \times C) + A.$$

What do you think?

(42) Vivian is right.

(43) How did Ann get from the line

$$A + (B \times C) = (B \times C) + A$$

to the line

$$A + (B \times C) = (C \times B) + A?$$

(43) She "switched across the multiplication sign";

$$(B \underset{\uparrow}{\times} C) + A$$

became

$$(C \underset{\uparrow}{\times} B) + A.$$

Consequently, Ann used the *commutative law for multiplication,* which says "you may multiply in either order,"

$$\square \times \triangle = \triangle \times \square.$$

(44) In question 43, what did Ann put into the $\square$? What did she put into the $\triangle$?

(44) What Ann *had* was

$$(B \times C) + A.$$

What she *wanted* was

$$(C \times B) + A.$$

Ann used the commutative law for multiplication:

$$\square \times \triangle = \triangle \times \square.$$

The equals sign is interpreted to mean that

$\square \times \triangle$ is just *another name* for $\triangle \times \square$.

If Ann substitutes B into the box and C into the triangle, she gets:

$$\boxed{B} \times \overset{\triangle}{C} = \overset{\triangle}{C} \times \boxed{B}$$

or

$$B \times C = C \times B.$$

This, of course, says that $C \times B$ is just *another name* for $B \times C$.

Consequently, in $(B \times C) + A$ we may substitute a new *name* for the *same* thing to get $(C \times B) + A$.

(45) Can you make up a derivation for this theorem?

Theorem: $(A + B) \times (C + D) = (D + C) \times (B + A)$

(45) **Here is one derivation:**

Theorem: $(A + B) \times (C + D) = (D + C) \times (B + A)$

Proof:

$$(A + B) \times (C + D) = (A + B) \times (C + D)$$

(We begin with a trivial identity.)

$$(A + B) \times (C + D) = (C + D) \times (A + B)$$

(We have used the commutative law for multiplication to get from the first line to the second. If we write the commutative law for multiplication as

$$\square \times \triangle = \triangle \times \square,$$

we have put $A + B$ into the box and $C + D$ into the triangle.)

$$(A + B) \times (C + D) = (D + C) \times (A + B)$$

(We got from the second line to the third by using the commutative law for addition.)

$$(A + B) \times (C + D) = (D + C) \times (B + A)$$

(We got from the third line to the fourth by again using the commutative law for addition.)

Q.E.D.

Q. E. D., of course, stands for the Latin *quod erat demonstrandum*. A rough English translation might be: "We have now proved what we set out to prove." It is traditionally written at the end of a proof, to indicate that the proof is now complete. The children in Madison Project classes consider it an exciting honor to be chosen to write Q. E. D.

(46) How many identities do you know that have names?

(46) **Here is (probably) the list:***

$\square = \square$	**Trivial**
$2 \times \square = 2 \times \square$	**Trivial**
$\square + \triangle = \triangle + \square$	**Commutative law for addition (CLA)**
$\square \times \triangle = \triangle \times \square$	**Commutative law for multiplication (CLM)**
$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$	**Distributive law (DL)**
$\square \times 1 = \square$	**Law for one (L1)**
$\square \times 0 = 0$	**Multiplication law for zero (MLZ)**

---

* The following abbreviations will be used in derivations:
ALA (associative law for addition), ALM (associative law for multiplication), ALZ (addition law for zero), MLZ (multiplication law for zero), CLA (commutative law for addition), CLM (commutative law for multiplication), DL (distributive law), L1 (law for one), CN (changing names), L Opp (law of opposites), CS ("changing signs").

$$\square + 0 = \square$$ **Addition law for zero (ALZ)**

$$(\square + \triangle) + \bigtriangledown = \square + (\triangle + \bigtriangledown)$$

Associative law for addition (ALA)

$$(\square \times \triangle) \times \bigtriangledown = \square \times (\triangle \times \bigtriangledown)$$

Associative law for multiplication (ALM)

(47) What is the longest list of identities that you can make up?

(47) This list should include all of the identities mentioned in answer to question 46, plus many more. Although you cannot predict what the students will come up with, the "many more" *might* include these:

$$\square + \square = 2 \times \square$$

$$(\square + \square) + \square = 3 \times \square$$

$$(5 \times \square) + (3 \times \square) = 8 \times \square$$

$$\frac{\square}{\square} = 1$$

$$(0 \times \square) + 0 = 0$$

$$(0 \times \square) + \square = \square$$

$$3 \times \square = \square \times 3$$

$$4 \times \square = \square \times 4$$

$$\square + 35 = (1 \times \square) + 34 + \tfrac{1}{6} + \tfrac{2}{3} + \tfrac{1}{6}^{*}$$

$$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$$

$$(\square + 5) \times (\square + 5) = (\square \times \square) + (10 \times \square) + 25$$

$$\square + \square - \square = \square$$

$$(5 \times \square) + (2 \times \square) + (103 \times \square) + (\tfrac{1}{2} \times \square) = (6 \times \square) + (\tfrac{1}{4} \times \square) + (\tfrac{1}{4} \times \square) + (4 \times \square) + (2 \times \square) + (98 \times \square)$$

(48) Can you shorten this list?

(a) $\square + \triangle = \triangle + \square$

(b) $\square \times \triangle = \triangle \times \square$

(c) $(A + B) \times (C + D) = (D + C) \times (B + A)$

(48) Delete statement (c), as proved in question 45.

---

* This identity was made up by a fifth-grade girl who hoped to trick her opponents into challenging it in the belief that it was not correct. This appears in one of the tape-recorded lessons.

chapter 29 / *Pages 56–57 of Student Discussion Guide*

DEBBIE'S LIST*

This lesson has three main purposes:

(a) To lead children to make up the *distributive law* as a generalization from instances.

(b) To provide further experience with *list shortening,* i.e., with *logical implication* and with *identities.*

(c) To give the children experience with the important *pattern* of the distributive law so that they may use it correctly and with confidence.

Chapter 29
DEBBIE'S LIST

[page 56]

(1) Last night Debbie memorized this list of identities.

(a) $A \times (B + 1) = (A \times B) + (A \times 1)$

(b) $A \times (B + 1) = (A \times B) + 1$

(c) $A \times (B + 2) = (A \times B) + (A \times 2)$

(d) $A \times (B + 7) = (A \times B) + (A \times 7)$

(e) $A \times (B + 8) = (A \times B) + (A \times 8)$

(f) $A \times (B + 9) = (A \times B) + 9$

(g) $\square \times (\triangle + 3) = (\square \times \triangle) + (\square \times 3)$

(h) $\square \times (\triangle + 1) = (\square \times \triangle) + (\square \times 1)$

(i) $\square \times (\triangle + 1) = (\square \times \triangle) + 1$

What do you think of Debbie's list?

ANSWERS AND COMMENTS

(1) There are three things "wrong" with Debbie's list:

(a) Items (b), (f), and (i) are not identities. (These problems are used to show the children what the distributive law is not.)

(b) Some identities or pseudoidentities *are listed twice,* but with different notation.

Specifically, (a) and (h) are identical, but are merely *written* so as to look *different.* (If you recall the meaning of $\square$, $\triangle$, *A, B,* etc., you see that these are actually the same identity.) Also, the pseudoidentities (b) and (i) are the same, but have been written so as to *look* different. [These problems are used to give the children more insight into the *pattern* of the distributive law, to strengthen their understanding of the basic concept of variable (or placeholder, or pronumeral), and to start them on the gradual transition from $\square$ and $\triangle$ to *A, B, a, b, x, y,* etc.]

(c) The list is unnecessarily long. In fact, after the wrong "identities" [items (b), (f), and (i)] are deleted, *all of the others can be replaced by the single identity*

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown).$$

At first your children may be impressed with Debbie's energy and accomplishment. But as they think about Debbie's list, the brighter children will begin to have some (well-founded) doubts.

Hopefully, the children will discover for themselves the identity that replaces Debbie's list.

A tape recording or sound film of this activity is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

The following questions work through the ideas of question 1 in detail.

(2) Joe says Debbie certainly did a good job. Do you agree?

(2) **No, for the reasons mentioned in answer to question 1.**

(3) Al says that Debbie wasted her time. What do you think?

(3) **Yes. See the answer to question 1.**

Questions 2 and 3 serve, among other things, to see if the children are being *honest*, and *thinking for themselves*, or if they are merely telling us what they think we want to hear. The hypocritical answer is, of course, to say that since Debbie did school work, she was being a good girl and spending her time wisely. The thinking-for-yourself answer involves asking if she might not have found something better to do, such as discovering

$$\square \times (\triangle + \triangledown) = (\square \times \triangle) + (\square \times \triangledown),$$

or going roller skating.

(4) Ellen says that some identities are listed twice. What do you think?

(4) **Yes. Item (a) is the same as (h), and item (b) is the same as (i).**

(5) Ellen claims she can cross two statements off of Debbie's list because they are repetitions of other statements on the list. Do you agree?

(5) **Yes. See the answer to questions 4 and 1.**

(6) Mike says that some statements on Debbie's list are not really identities at all. What do you think?

(6) **Mike is right. Items (b), (f), and (i) are *not* identities.**

(7) Have you been able to shorten Debbie's list?

(7) **Presumably, by now. But there's still more to come!**

(8) Did Debbie really waste her time, or not?

(8) **Hopefully by now more of the children are getting interested, and are beginning to wonder.**

(9) Vivian says that she thinks **one** identity could replace Debbie's entire list. Can you make up **one** identity that can replace Debbie's whole list?

(9) **There is one identity that will replace the whole list, namely:**

$$\square \times (\triangle + \triangledown) = (\square \times \triangle) + (\square \times \triangledown).$$

This identity is known as the distributive law. It can also be written as

$$A \times (B \times C) = (A \times B) + (A \times C),$$

although at this stage it is probably better to use the $\square$, $\triangle$, $\triangledown$ version, on the grounds that your children are (probably) not yet ready for too much use of *letters* in place of boxes.

(10) John says he thinks that this identity

$$\square \times (\triangle \times \triangledown) = (\square \times \triangle) \times (\square \times \triangledown)$$

can replace Debbie's whole list. Do you agree?

(10) **No. John's identity is wrong (but it's almost right). Hopefully, your children will see what is wrong with John's list and be able to correct it.**

(11) Bob says that John's identity is wrong. Bob says it is not really an identity. What do you think?

(11) **Bob is right. The identity is wrong.**

(12) A girl named Lou says that this identity

$$\square \times (\triangle + \triangledown) = (\square \times \triangle) + (\square \times \triangledown)$$

can replace Debbie's whole list. Do you agree?

(12) **Lou is right.**

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(13) Mary says that Lou's "identity" isn't really an identity. Who is right?

(13) **Lou is right.**

(14) Can you shorten Debbie's list?

(14) **Yes. It can be replaced by the single identity**

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown).$$

This is analogous to replacing the entire list

$$\begin{aligned} 3 \times \square &= \square \times 3 \\ 4 \times \square &= \square \times 4 \\ 5 \times \square &= \square \times 5 \\ 6 \times \square &= \square \times 6 \\ &\vdots \end{aligned}$$

by the single identity

$$\triangle \times \square = \square \times \triangle.$$

Again, it is roughly analogous to replacing the list

Bob's sister Joan goes to school.
Bob's sister Harriet goes to school.
Bob's sister Ellen goes to school.
Bob's brother Francis goes to school.
Bob's brother Walter goes to school.

by the single statement

All of Bob's brothers and sisters go to school.

(You could argue over this last example. This is almost always possible where examples are chosen from *everyday* life and language. Such examples lack the abstract precision of mathematical examples, and tend to involve extraneous, irrelevant complications, e.g., the last statement assumes that Bob has only three sisters and two brothers.)

(15) Did Debbie spend her time wisely?

(15) **Perhaps, by now, the answer is "no."**

chapter 30 / *Pages 57–60 of Student Discussion Guide*

LEX'S IDENTITY

This chapter continues the work on identities and list shortening. It is rather similar to Chapter 29.

Chapter 30
LEX'S IDENTITY

[page 57]

(1) What is the longest list of identities that you know?

(2) Some identities are so famous that they have names. Do you know any identities that have names?

(3) Larry says that this identity

$$\triangle + \square = \square + \triangle$$

is known as the **commutative law for addition.** What do you suppose this identity

$$\triangle \times \square = \square \times \triangle$$

is called?

(4) Ernie says that Lou's identity

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

is known as the **distributive law.** Do you think Ernie is right?

(5) One identity is known as the **law for one.** What do you suppose it is?

(6) One identity is known as the **addition law for zero.** Which identity do you think it is?

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(7) Then there is an identity that is called the **multiplication law for zero.** Which one do you think it is?

(8) Do you think this identity

$$A + (B \times C) = (C \times B) + A$$

has a name? Can you guess why?

ANSWERS AND COMMENTS

(1) **See answers to problems 46 and 47 in Chapter 28.**

(2) **See answers to problem 46 in Chapter 28.**

(3) **As you might guess (by analogy), it is called the *commutative law for multiplication.***

(4) **Ernie is right. (The question, of course, is largely rhetorical.)**

(5) $\square \times \mathbf{1} = \square$ $\left(\textbf{or } \mathbf{1} \times \square = \square\right)$

You should choose one way or the other of writing this, and thereafter be consistent. If you choose to call

$$\square \times 1 = \square$$

the *law for one*, then, of course,

$$1 \times \square = \square$$

is a theorem obtained from the *law for one* and the *commutative law for multiplication*.

(6) $\square + \mathbf{0} = \square$

(7) $\mathbf{0} \times \square = \mathbf{0}$ $\left(\textbf{or } \square \times \mathbf{0} = \mathbf{0}\right)$

Again, choose one way of writing this identity, and thereafter, if possible, be consistent.

(8) **It does not have a name. This identity does not really need a name since it can be derived from the commutative law for multiplication and the commutative law for addition.**

(9) What do we mean by a **trivial** identity?

(9) One which is *exactly* the same on both sides of the equals sign, such as:

$$\square + (3 \times \triangle) = \square + (3 \times \triangle).$$

(10) Can you make a derivation for this identity?

$A \times (B + 5) = (5 \times A) + (B \times A)$

(10) Here is one possible derivation:

Theorem: $A \times (B + 5) = (5 \times A) + (B \times A)$

Proof:

$A \times (B + 5) = A \times (B + 5)$ **A "trivial" identity; therefore guaranteed**

DL

$A \times (B + 5) = (A \times B) + (A \times 5)$

CLA

$A \times (B + 5) = (A \times 5) + (A \times B)$

CLM

$A \times (B + 5) = (5 \times A) + (A \times B)$

CLM

$A \times (B + 5) = (5 \times A) + (B \times A)$

Q.E.D.

(11) Here is part of a derivation of the identity

$A \times (3 + W) = (W \times A) + (3 \times A).$

See if you can fill in the rest.

Theorem: $A \times (3 + W) = (W \times A) + (3 \times A)$

Proof:

$A \times (3 + W) =$ ____________ Distributive law

$A \times (3 + W) =$ ____________ Commutative law for addition

$A \times (3 + W) =$ ____________

________ $= (W \times A) + (A \times 3)$ Reason?

________ $=$ ____________ Reason?

Q.E.D.

(11) Theorem: $A \times (3 + W) = (W \times A) + (3 \times A)$

Proof: $A \times (3 + W) = A \times (3 + W)$

DL

$A \times (3 + W) = (A \times 3) + (A \times W)$

CLA

$A \times (3 + W) = (A \times W) + (A \times 3)$

CLM

$A \times (3 + W) = (W \times A) + (A \times 3)$

CLM

$A \times (3 + W) = (W \times A) + (3 \times A)$

Q.E.D.

(12) Do you know what "Q.E.D." means? What language is it?

(12) *Quod erat demonstrandum* (which was to be proved). It is Latin.

(13) Debbie says that

"2" is just another name for "1 + 1",
and "3" is just another name for "2 + 1",
and "4" is just another name for "3 + 1",
and so on.

Debbie has added these statements to her basic list of identities.

$2 = 1 + 1$
$3 = 2 + 1$
$4 = 3 + 1$
$\vdots$

She calls this **changing names.** What do you think?

(13) These "changing-names" rules are an important addition. They should be part of the list of axioms. When combined with the identity axioms, they serve to introduce *all the facts of arithmetic*.

Of course, in earlier grades the children have learned all the basic arithmetic facts and procedures, such as $8 + 3 = 11$, $2 \times 2\frac{1}{2} = 5$, $2 + 2 = 4$, and so on.

Now, at this more sophisticated stage, we are interested mainly (for the moment) in formal deduction, and we are therefore seeking *the shortest list of statements from which all other algebraic and arithmetical statements can be derived.* Debbie's "changing-names" list (the recursive definition of the numerals) is an important part of this short list.

(14) See if you can make a derivation for this theorem.

Theorem: $2 \times 3 = 3 + 3$

(14) Here is one derivation ("changing names"(CN) refers to the list in question 13):

Theorem: $2 \times 3 = 3 + 3$

Proof:

$$\begin{aligned} 2 \times 3 &= 2 \times 3 \\ &\quad \text{CN} \\ 2 \times 3 &= (1 + 1) \times 3 \\ &\quad \text{CLM} \\ 2 \times 3 &= 3 \times (1 + 1) \\ &\quad \text{DL} \\ 2 \times 3 &= (3 \times 1) + (3 \times 1) \\ &\quad \text{L1, used twice} \\ 2 \times 3 &= 3 + 3 \end{aligned}$$

Q.E.D.

(15) Which of these are identities?

(a) $\square + (\square \times 1) = (\square + \square) \times (\square + 1)$

(b) $\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$

(c) $\square + \square = 2 \times \square$

(d) $\square + \square + \square = 3 \times \square$ [page 59]

(e) $A \times (B + 5) = (A \times B) + (A \times 5)$

(f) $A \times (B + 7) = (A \times B) + 7$

(15) (a) Not an identity
(b) An identity (in fact, the distributive law)
(c) An identity
(d) An identity
(e) An identity
(f) Not an identity

(16) Can you shorten this list without really losing anything?

(a) $A \times 3 = 3 \times A$

(b) $A \times 7 = 7 \times A$

(c) $A \times 15 = 15 \times A$

(d) $A \times 100 = 100 \times A$

(e) $A \times 1{,}000{,}000 = 1{,}000{,}000 \times A$

(f) $A \times 1980 = 1980 \times A$

(g) $A \times B = B \times A$

(h) $\square \times \triangle = \triangle \times \square$

(i) $\nabla \times \square = \square \times \nabla$

(16) The entire list can be replaced by the single identity of item (h).

(17) Is this an identity?

$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$

(17) Surprise! It is!

Can you complete each statement so that it will be an identity?

(18) $(\square + 4) \times (\square + 4)$
$= (\square \times \square) + \_\_\_\_\_\_\_\_\_\_\_\_$

(18) $(8 \times \square) + 16$

(19) $(\square + 5) \times (\square + 5)$
$= (\square \times \square) + \_\_\_\_\_\_\_\_\_\_\_\_$

(19) $(10 \times \square) + 25$

(20) $(\square + 11) \times (\square + 11)$
$= (\square \times \square) + \_\_\_\_\_\_\_\_\_\_\_\_$

(20) $(22 \times \square) + 121$

(21) $(\square + 7) \times (\square + 7)$
$= (\square \times \square) + \_\_\_\_\_\_\_\_\_\_\_\_$

(21) $(14 \times \square) + 49$

(22) Can you shorten this list somehow?

(a) $(\square + 2) \times (\square + 2)$
$= (\square \times \square) + (4 \times \square) + 4$

(b) $(\square + 6) \times (\square + 6)$
$= (\square \times \square) + (12 \times \square) + 36$

(c) $(\square + 10) \times (\square + 10)$
$= (\square \times \square) + (20 \times \square) + 100$

(d) $(\square + 12) \times (\square + 12)$
$= (\square \times \square) + (24 \times \square) + 144$

(e) $(\square + 13) \times (\square + 13)$ [page 60]
$= (\square \times \square) + (26 \times \square) + 169$

(f) $(\square + 8) \times (\square + 8)$
$= (\square \times \square) + (16 \times \square) + 64$

(23) Lex says he can write one identity that can replace the entire list in question 22.

Can you?

(22) This whole list can be replaced by the single identity

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$$

or, equivalently (take your choice), by

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + [(2 \times \triangle) \times \square] + (\triangle \times \triangle).$$

(23) This is possible. See answer to question 22.

chapter 31 / ***Pages 60–62 of Student Discussion Guide***

NAMES FOR NUMBERS

The purpose of this chapter is to give a specific interpretation of the equals sign: namely, that

$$2 + 2 = 4$$

means that "4" is a *name* for a number, and "2 + 2" is *another name for that same number.* This eliminates many complicated circumlocutions since we need only the single rule of substitution: *in place of one name, some other name for the same thing* may be substituted.

This interpretation is very valuable in making derivations.

Chapter 31
NAMES FOR NUMBERS

ANSWERS AND COMMENTS

[page 60]

(1) (2 + 5) is a name for some number. What is **another** name for this same number?

(1) Here are a few:

$6 + 1$
$\frac{14}{2}$
7
$2 \times 3\frac{1}{2}$
$6\frac{3}{4} + \frac{1}{4}$
$1 + 1 + 1 + 1 + 1 + 1 + 1$

(2) How many names can you give for the number (7 + 8)?

(2) Here are a few:

$\frac{30}{2}$
$7\frac{1}{2} \times 2$
$8 + 7$
$14 + 1$
15
$18 - 3$

(3) Jerry says that the meaning of the sign "=" is that you have **two names for the same number.** For example,

$$\frac{2}{3} \times \frac{3}{4} = \frac{1}{4} + \frac{1}{4}$$

really means that

$$\frac{2}{3} \times \frac{3}{4}$$

is a name for some number, and

$$\frac{1}{4} + \frac{1}{4}$$

is a name for **this same number.**

Do you agree?

(3) Jerry is right. (This is the interpretation we wish to recommend.)

(4) If we had a close-up photo of Pablo Casals, and a distance shot of him, and a profile shot, and a picture from the rear, we would have photos of how many people?

(4) *One person*

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(5) If we have these names

$7 + 5$	$5 + 8$
$18 - 6$	$11 + 7$
$18 \times \frac{2}{3}$	$21 - 9$
$\frac{1}{4} \times 20$	2×6,

how many different numbers do we have?

(5) There are four different numbers mentioned here, namely: 12, 13, 18, 5.

(6) Joe says that

$$\frac{1}{3} \times \frac{3}{8}$$
$$2 \times \frac{1}{16}$$
$$1 - \frac{7}{8}$$

are all names for the **same** number. Do you agree?

(6) Joe is right; the number more commonly goes by the name $\frac{1}{8}$.

(7) Mary says that

$$\frac{1}{2} + \frac{1}{2}$$
$$2 - 1$$
$$7 \times \frac{1}{7}$$
$$21 \times \frac{1}{21}$$
$$\frac{1}{4} + \frac{3}{4}$$
$$\frac{7}{8} + \left(2 \times \frac{1}{16}\right)$$

are all names for the **same** number. Do you agree?

(7) Mary is right.

(8) How many different numbers are there in this list?

(a) $3 + 2$
(b) $7 \times \frac{5}{7}$
(c) $\frac{5}{3} \times 3$
(d) $\frac{21}{8} \times 8$
(e) 3×7
(f) 5
(g) $28 - 2$
(h) $25 - 4$
(i) $^{+}22 + {}^{-}1$
(j) $^{+}20 - {}^{-}1$

(8) Three numbers, namely: 5, 21, 26.

(9) In question 8, which **number** is represented on the list by the **most names?** How many names for this number are there on the list?

(9) The number 21 is represented by five different names.

(10) Jerry says that we may always replace one name for a number by any other name for that same number. Do you agree?

(10) Yes

(11) "Elizabeth Wilson sits in the back row." Can you replace "Elizabeth Wilson" by some other name for the same girl?

(11) Yes, for example: "Betty Wilson sits in the back row."

(12) Marie says that the identity **[page 62]**

$$\triangle + \square = \square + \triangle$$

really means that

$$3 + 4$$

is a name for a number, and

$$4 + 3$$

is another name for that same number.

Do you agree?

(12) Yes

(13) Al says that

$$a + (b + c) = a + (c + b)$$

can be derived by starting with the trivial identity

$$a + (b + c) = a + (b + c)$$

and then using the identity

$$\square + \triangle = \triangle + \square,$$

which says that

$$b + c$$

is a name for a number, and

$$c + b$$

is another name for that same number.

Consequently, Al writes

$$a + (b + c) = a + (\quad).$$

Now, in the space, he puts another name for the number.

Which name does he use?

(13) $c + b$

(14) If you have the list of identities

(a) $\square + \triangle = \triangle + \square$

(b) $\square \times \triangle = \triangle \times \square$

(c) $(a + b) \times (c + d) = (d + c) \times (b + a)$,

can you make up a derivation for (c) by starting with the trivial identity

$$(a + b) \times (c + d) = (a + b) \times (c + d)?$$

(14) Here is one derivation:

Theorem: $(a + b) \times (c + d) = (d + c) \times (b + a)$

Proof:
$$\begin{aligned}
(a + b) \times (c + d) &= (a + b) \times (c + d) \\
&\qquad \text{CLM} \\
(a + b) \times (c + d) &= (c + d) \times (a + b) \\
&\qquad \text{CLA} \\
(a + b) \times (c + d) &= (d + c) \times (a + b) \\
&\qquad \text{CLA} \\
(a + b) \times (c + d) &= (d + c) \times (b + a)
\end{aligned}$$

Q. E. D.

chapter 32 / *Pages 63–66 of Student Discussion Guide*

THE AXIOMS*

Chapter 32
THE AXIOMS

[page 63]

(1) What is the longest list of identities that you can think of?

(2) Jerry made up this list of identities:

$\square \times 0 = 0$

$\square + 0 = \square$

$\square \times 1 = \square$

$\square \times 1 = 1 \times \square$

$\square \times 2 = 2 \times \square$

$\square \times 3 = 3 \times \square$

$\square \times 4 = 4 \times \square$

$\square \times 5 = 5 \times \square$

$\vdots$

$\square + 1 = 1 + \square$

$\square + 2 = 2 + \square$

$\square + 3 = 3 + \square$

$\square + 4 = 4 + \square$

$\vdots$

$\square \times (\square + 1) = (\square \times \square) + \square$

$\square \times (\square + 2) = (\square \times \square) + (\square \times 2)$

$\square \times (\triangle + 1) = (\square \times \triangle) + (\square \times 1)$

$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$

$\square \times \triangle = \triangle \times \square$

$(\square + \triangle) \times (\square + \triangle) = (\square \times \square)$
$+ [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$

$A + B = B + A$

$A \times B = B \times A$

$A \times (B + C) = (A \times B) + (A \times C)$

$(A + B) \times (A + B)$
$= (A \times A) + [(B + B) \times A] + (B \times B)$

ANSWERS AND COMMENTS

(1) **This list will depend upon your class.**

(2) **Jerry's list is a good one. However, it could be greatly shortened.**

A tape recording or sound film of this activity is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

$\square + \square = 2 \times \square$ **[page 64]**

$\square + \square + \square = 3 \times \square$

$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$

What do you think of Jerry's list?

(3) Suppose you spent a whole year making up the longest list of identities you could.

And then suppose you spent the following year shortening this list by every method you could think of.

What do you think the **final** list would be?

(3) An answer is not given here in order not to limit the freedom of your class discussion too severely.

(4) Doris says she thinks the final list would look like this:

(a) $\square = \square$

(b) $\square + \triangle = \triangle + \square$

(c) $\square \times \triangle = \square \times \triangle$

(d) $\square \times (\triangle + \bigtriangledown)$
$= (\square \times \triangle) + (\square \times \bigtriangledown)$

(e) $\square \times 0 = 0$

(f) $\square + 0 = \square$

(g) $\square \times 1 = \square$

(h) $\frac{\square}{\square} = 1$

(i) $\square \times (\triangle + 5) = (\square \times \triangle) + (\square \times 5)$

(j) $\square \times 1 = 1 \times \square$

(k) $\square \times 0 = 0 \times \square$

What do you think of Doris' list?

(4) The last three identities on Doris' list are clearly not necessary. (That is to say, they are theorems rather than axioms. Items (j) and (k) are instances of the commutative law for multiplication and item (i) is merely a special case of the *distributive law*.)

(5) Don says he thinks the final list would look like this:

(a) $\square = \square$

(b) $\square + \triangle = \triangle + \square$

(c) $\square \times \triangle = \triangle \times \square$

(d) $\square \times (\triangle + \bigtriangledown)$
$= (\square \times \triangle) + (\square \times \bigtriangledown)$

(e) $(\triangle + \bigtriangledown) \times \square$ **[page 65]**
$= (\triangle \times \square) + (\bigtriangledown \times \square)$

(5) Don's list contains at least two unnecessary identities. Item (e) follows immediately from the distributive law and the commutative law for addition. Item (h) is, in fact, a theorem, although its derivation is somewhat complicated.

(f) $\square \times 1 = \square$

(g) $$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \quad\vdots \end{array}\right\} \text{Changing names}$$

(h) $\square + \square = 2 \times \square$

What do you think?

(6) Ellen says she can find at least two theorems on Don's list. Can you?

(6) Yes. See answer to question 5.

(7) What does Ellen mean by saying that there are two theorems on Don's list?

(7) See answer to question 5.

Remember, when *shortening lists* by erasing as many identities as possible without *really* losing anything (i.e., by using generalization and implication), then:

> the identities that remain on the final shortest list are called *axioms*,

and

> the identities that are erased because they are implied by the axioms are called *theorems*.

For example: if we take the list

(a) $\square + \triangle = \triangle + \square$
(b) $\square \times \triangle = \triangle \times \square$
(c) $A + (B \times C) = (C \times B) + A$

and shorten it without losing anything by discarding the third statement, then items (a) and (b), which remain on the final list, are called *axioms*. Item (c), which is implied by the commutative laws for addition and multiplication (the other two statements on the list), can be deleted. It is, consequently, called a *theorem*.

(8) Is Ellen right?

(8) Yes. See answers to questions 5, 6, and 7.

(9) How does Don's list take care of zero?

(9) No. Don's list does *not* provide for zero. This is an omission, or defect. In fact, there are two kinds of errors on Don's list—some identities *are* there but *should not be,* while, on the other hand, some identities *are not* there, but *should be*.

(10) Daria made up this list:

(a) $\square = \square$
(b) $\square + \triangle = \triangle + \square$
(c) $\square \times \triangle = \triangle \times \square$
(d) $\square \times (\triangle + \bigtriangledown)$
$= (\square \times \triangle) + (\square \times \bigtriangledown)$
(e) $\square \times 0 = 0$
(f) $\square + 0 = \square$

(10) Yes. Daria's list is probably the best list that your students can make at this stage of their knowledge. You may (or they may) possibly have introduced the *associative* laws. If so, they should be included on the list. In any event, this list will be extended somewhat in later lessons.

(g) $\square \times 1 = \square$

(h) $$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \quad\;\vdots \end{array}\right\} \text{Changing names}$$

Is Daria's list right?

(11) What do we mean by an **axiom**?

(12) Cynthia made up this list:

(a) $\square = \square$

(b) $\square + \triangle = \triangle + \square$

(c) $\square \times \triangle = \triangle \times \square$

[page 66]

(d) $\square \times (\triangle + \bigtriangledown)$
$= (\square \times \triangle) + (\square \times \bigtriangledown)$

(e) $\square \times 0 = 0$

(f) $\square + 0 = \square$

(g) $\square \times 1 = \square$

(h) $$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \quad\;\vdots \end{array}\right\} \text{Changing names}$$

(i) $(\square + \triangle) \times (\square + \triangle) = (\square \times \square)$
$+ [(\triangle + \triangle) \times \square] + (\triangle \times \triangle)$

What do you think of Cynthia's list?

(11) See the discussion following the answer to question 7.

(12) Cynthia's list is a very good one, but, as a matter of fact, the last identity is actually a ***theorem***. (It is not too hard to make a derivation to show this, but do not be misled into thinking that it's easy. Indeed, $\square \times 0 = 0$ is also a theorem; the proof of this, however, is more difficult.)

chapter 33 / *Pages 66–67 of Student Discussion Guide*

DERIVATIONS

Derivations are the very heart of algebra. Every *true* statement in algebra is true either because it is an *axiom*, and it is *agreed* to label it true, or because it is a *theorem*, which can be derived from the axioms. This might be called the logician's view of mathematical structure.

From a pedagogical point of view, *derivations* provide the most effective route to the mastery of manipulative technique. They have the great advantage that manipulative technique grows naturally out of experience with derivations, instead of appearing magically and incomprehensibly cloaked in "rules for removing parentheses" and such like.

Thus, as frequently happens, what is mathematically and logically desirable is also pedagogically desirable. We throw out the old-fashioned rules for removing parentheses, rules for multiplying binomials, and so on, and achieve the same end in a superior way, mathematically and pedagogically, by introducing *derivations.*

For a teacher, this usually takes some getting used to, but after a little experience, most teachers (and children) find that derivations are fun. Do not be dismayed if you encounter some difficulties at first. They exist, but they are not insurmountable.

Chapter 33
DERIVATIONS

ANSWERS AND COMMENTS

[**page 66**]

(1) We have a special list of identities. Do you know what they are called?

(1) **Axioms (i.e., the shortest list of identities from which all others can be derived).**

Actually, we shall want to include on our shortest list one set of statements *that are not identities*, namely the changing-names statements (recursive definition of numerals):

$$\left.\begin{aligned} 1+1&=2\\ 2+1&=3\\ 3+1&=4\\ &\vdots \end{aligned}\right\}\ \text{Changing names}$$

We regard changing names as embodying the essential notion of *closure, but it is better not to mention closure during the first year's study of algebra.* We occasionally build gradually toward this idea by asking if certain equations can be solved within a restricted set of numbers (for example, the set of integers).

(2) What is "special" about this list?

(2) **It is the shortest list of statements from which all of the identities of algebra can be derived.**

Actually, since this is, for the students, a new and growing science, their list of axioms will be subject to occasional modifications in the light of new developments. If they do not yet know about the *associative* laws, they will want to add these later; similarly with the laws relating to *subtraction*, and so on.

(3) See if you can write down this list.

(3) Your children *may* write this list:

$\square + \triangle = \triangle + \square$	**Commutative law for addition**
$\square \times \triangle = \triangle \times \square$	**Commutative law for multiplication**
$\square \times 1 = \square$	**Law for one**
$\square \times 0 = 0$	**Multiplication law for zero**
$\square + 0 = \square$	**Addition law for zero**
$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$	**Distributive law**

They *may* also add

$$\frac{\square}{\square} = 1.$$

(Although this has a somewhat doubtful status, it is very valuable in arithmetic, and our students usually like to include it.)

They should probably include

$$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \quad\vdots \end{array}\right\} \text{Changing names}$$

where, as usual, the final three dots mean that *the list goes on forever.*

They might also include $\square = \square$.

If you have included the associative laws, your children will presumably list them as axioms (which, indeed, they actually are):

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

Associative law for addition (ALA)

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$

Associate law for multiplication (ALM)

It is assumed, however, that you have *not* yet mentioned the laws involving subtraction or oppositing; therefore, these identities are omitted from the list at this point. The list of axioms, like the list of wonder drugs, is subject to change with the appearance of new discoveries.

(4) Do you know the name of each identity?

(4) The names are included in the answer to question 3.

(5) Can you make a derivation for this identity?

$$\square + \square = 2 \times \square$$

(5) **Here is one of several possible derivations:**

Theorem: $\square + \square = 2 \times \square$

$$
\begin{array}{rl}
\textbf{Proof:}\quad 2 \times \square = 2 \times \square & \\
 & \text{CN} \\
(1 + 1) \times \square = 2 \times \square & \\
 & \text{CLM} \\
\square \times (1 + 1) = 2 \times \square & \\
 & \text{DL} \\
(\square \times 1) + (\square \times 1) = 2 \times \square & \\
 & \text{LI} \\
\square + (\square \times 1) = 2 \times \square & \\
 & \text{LI} \\
\square + \square = 2 \times \square & \\
\text{Q. E. D.} &
\end{array}
$$

(6) Can you make a derivation for this identity?

$$A + (B \times C) = (C \times B) + A$$

(6) **Here is one possible derivation:**

Theorem: $A + (B \times C) = (C \times B) + A$

$$
\begin{array}{rl}
\textbf{Proof:}\quad A + (B \times C) = A + (B \times C) & \\
 & \text{CLA} \\
A + (B \times C) = (B \times C) + A & \\
 & \text{CLM} \\
A + (B \times C) = (C \times B) + A & \\
\text{Q. E. D.} &
\end{array}
$$

This illustrates an important point. After the difficult derivation in problem 5, the students are given a far easier one to tackle. You might call this the "chocolate sundae after you finish your spinach" principle. Problems arranged in an order of increasing difficulty intimidate even the hardy among our children, and could reasonably be expected to sour the outlook of even the otherwise optimistic. Who wants a world that can only get worse?

See if you can make a derivation for each of these.

(7) $A \times (B + C) = (C + B) \times A$

(7) **Theorem:** $A \times (B + C) = (C + B) \times A$

$$
\begin{array}{rl}
\textbf{Proof:}\quad A \times (B + C) = A \times (B + C) & \\
 & \text{CLA} \\
A \times (B + C) = A \times (C + B) & \\
 & \text{CLM} \\
A \times (B + C) = (C + B) \times A & \\
\text{Q. E. D.} &
\end{array}
$$

(8) $(A + B) \times (C + D)$
$= (A \times C) + (A \times D) + (B \times C) + (B \times D)$

(8) **Theorem:** $(A + B) \times (C + D) = (A \times C) + (A \times D) + (B \times C) + (B \times D)$

Proof:

$$\begin{aligned}
(A+B)\times(C+D) &= (A+B)\times(C+D) \\
(A+B)\times(C+D) &= [(A+B)\times C] + [(A+B)\times D] && \text{DL} \\
(A+B)\times(C+D) &= [C\times(A+B)] + [D\times(A+B)] && \text{CLM (used twice)} \\
(A+B)\times(C+D) &= (C\times A)+(C\times B)+[D\times(A+B)] && \text{DL} \\
(A+B)\times(C+D) &= (C\times A)+(C\times B)+(D\times A)+(D\times B) && \text{DL} \\
(A+B)\times(C+D) &= (A\times C)+(C\times B)+(D\times A)+(D\times B) && \text{CLM} \\
(A+B)\times(C+D) &= (A\times C)+(B\times C)+(D\times A)+(D\times B) && \text{CLM} \\
(A+B)\times(C+D) &= (A\times C)+(B\times C)+(A\times D)+(D\times B) && \text{CLM} \\
(A+B)\times(C+D) &= (A\times C)+(B\times C)+(A\times D)+(B\times D) && \text{CLM} \\
(A+B)\times(C+D) &= (A\times C)+(A\times D)+(B\times C)+(B\times D) && \text{CLA}
\end{aligned}$$

Q. E. D.

This derivation is presented on the assumption that you have not yet introduced the associative laws. (The children usually assume a nonexplicit, nonverbalized general associative law which—as long as it is never brought to their attention—can prove very serviceable at this stage in the course.)

(9) $2 \times 5 = 5 + 5$

(9) **Theorem:** $2 \times 5 = 5 + 5$

Proof:

$$\begin{aligned}
2 \times 5 &= 2 \times 5 \\
2 \times 5 &= (1 + 1) \times 5 && \text{CN} \\
2 \times 5 &= 5 \times (1 + 1) && \text{CLM} \\
2 \times 5 &= (5 \times 1) + (5 \times 1) && \text{DL} \\
2 \times 5 &= 5 + 5 && \text{L1 (used twice)}
\end{aligned}$$

Q. E. D.

Of course, a second way to prove this would be to take the identity

$$\square + \square = 2 \times \square,$$

which we already know to be valid (thanks to problem 5 above), and to insert 5 into each box.

(10) $(A + B) \times (A + B)$
$= (A \times A) + [(B + B) \times A] + (B \times B)$

(10) **The derivation is deliberately omitted in order not to spoil your fun in discovering these interesting derivations for yourself.**

[page 67]

(11) Try to complete the right-hand side so that the result will be an identity.

$(A + B + C) \times (D + E + F)$
$= (A \times D) +$ ______________________

(11) $(A \times E) + (A \times F) + (B \times D) + (B \times E) + (B \times F) + (C \times D) + (C \times E) + (C \times F)$

(12) Try to make a derivation for this identity.

$(A + B) \times (A + W)$
$= (A \times A) + [(B + W) \times A] + (B \times W)$

(12) **See problem 10.**

chapter 34 / *Pages 67–68 of Student Discussion Guide*

USING IDENTITIES

Identities provide an extremely valuable tool in arithmetic. We have found it far more satisfactory to get the identities in box-triangle notation *first*, and *thereafter* apply them to specific instances using numbers, rather than to work the other way around. Here, once again, what is *true to the nature of genuine mathematics* is also *especially suitable to effective pedagogy.*

Chapter 34
USING IDENTITIES

[page 67]

(1) Is this an identity?

$$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$$

(2) Jerry says we can multiply

$$4 \times 2\tfrac{1}{4}$$

as follows:

$$2\tfrac{1}{4} = \tfrac{9}{4}$$

$$4 \times \tfrac{9}{4} = \tfrac{36}{4} = 9$$

Is Jerry's method correct?

(3) Alan says we can multiply $4 \times 2\frac{1}{4}$ as follows:

$$4 \times 2 = 8$$

$$4 \times \tfrac{1}{4} = 1$$

$$\begin{array}{r} 8 \\ +\ 1 \\ \hline 9 \end{array}$$

Is Alan's method correct?

(4) Elizabeth says that Jerry used one side of the distributive–law identity and Alan used the other side.

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Which side did Jerry use?
Which side did Alan use?
Is Elizabeth right?

Multiply by two different methods.

(5) $8 \times 7\frac{1}{8} = ?$

ANSWERS AND COMMENTS

(1) **Yes. It is the distributive law.**

(2) **Yes.**

This is a method which many children use.

(3) **Yes.**

Relatively few fifth or sixth graders use this method, but *very* young children (*say*, first graders) often discover it for themselves and use it correctly before they have ever begun to receive formal instruction in working with fractions.

(4) **Elizabeth is right. Substitute as follows:**

$$\underset{\substack{\uparrow \\ 4}}{\square} \times (\underset{\substack{\uparrow \\ 2}}{\triangle} + \underset{\substack{\uparrow \\ \frac{1}{4}}}{\bigtriangledown}) = (\underset{\substack{\uparrow \\ 4}}{\square} \times \underset{\substack{\uparrow \\ 2}}{\triangle}) + (\underset{\substack{\uparrow \\ 4}}{\square} \times \underset{\substack{\uparrow \\ \frac{1}{4}}}{\bigtriangledown})$$

$$\boxed{4} \times (\overset{\triangle}{2} + \underset{\bigtriangledown}{\tfrac{1}{4}}) = (\boxed{4} \times \overset{\triangle}{2}) + (\boxed{4} \times \underset{\bigtriangledown}{\tfrac{1}{4}})$$

$$\mathbf{4 \times (2 + \tfrac{1}{4}) = (4 \times 2) + (4 \times \tfrac{1}{4})}$$

The left-hand side, which says *first* add $2 + \frac{1}{4}$ and *then* multiply by 4, corresponds to Jerry's method; the right-hand side, which says *first* multiply ($4 \times 2 = 8$ and $4 \times \frac{1}{4} = 1$) and *then* add ($8 + 1 = 9$), corresponds to Alan's method.

(5) **First method: $8 \times \frac{57}{8} = 57$**

Second method: $\mathbf{8 \times (7 + \tfrac{1}{8}) = (8 \times 7) + (8 \times \tfrac{1}{8})}$

$$\mathbf{= 56 + 1}$$

$$\mathbf{= 57}$$

	Problems 5 through 9 are, of course, applications of the distributive law.
(6) $6 \times 2\frac{1}{3} = ?$	(6) **See questions 1 through 4.**
(7) $16 \times 2\frac{1}{2} = ?$	(7) **$16 \times (2 + \frac{1}{2}) = 32 + 8 = 40$. See questions 1 through 4.**
(8) $9 \times 5\frac{2}{9} = ?$	(8) **$9 \times (5 + \frac{2}{9}) = 45 + 2 = 47$. See questions 1 through 4 and also question 5.**
(9) $7 \times 2\frac{3}{7} = ?$	(9) **Similar to the preceding questions. Here are the two methods, each written out in full detail:** **First method (*add* first, then multiply): $7 \times \frac{17}{7} = 17$.** **Second method (*multiply* first, then add): $7 \times (2 + \frac{3}{7}) = 14 + 3 = 17$.** **The distributive law assures us that these two will always be equal.**
(10) Is this an identity? $\frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$	(10) **Yes. (You can argue if *B* or *D* is zero.)** This is another suggestion (actually, the merest hint) of how *you* can combine algebra and arithmetic to the mutual advantage of both subjects.

chapter 35 / *Pages 68–73 of Student Discussion Guide*

GRAPHS OF TRUTH SETS

This chapter resumes the graphing of linear functions (which are usually of the form △ = (___ × □) + ___, and always yield straight-line pictures for the truth set), starting sometimes with the equation and asking for the graph and starting in other cases with the graph and asking for the equation.

The chapter also introduces conic sections, *where the picture of the truth set does not form a straight line.*

Chapter 35
GRAPHS OF TRUTH SETS

ANSWERS AND COMMENTS

[page 68]

(1) Can you use a table to show the truth set for

△ = (2 × □) + 3?

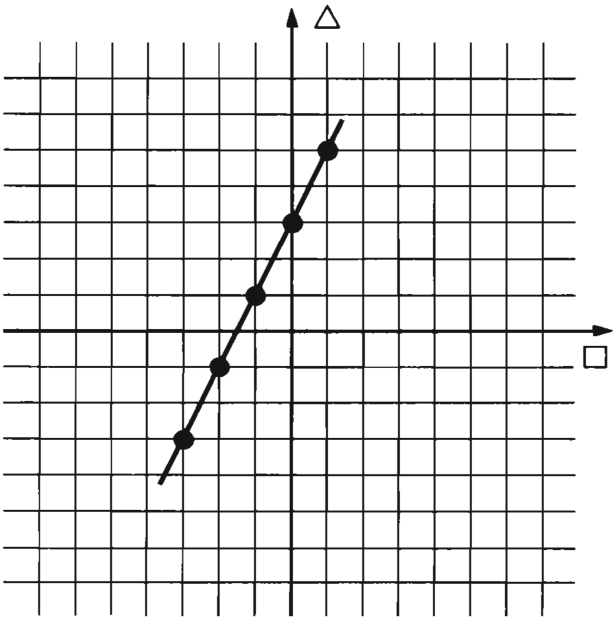
(1) **Here are some possible values to enter in a table:**

□	△
0	3
1	5
2	7
3	9
4	11
⋮	⋮

(2) Can you use a graph to show the truth set for

△ = (2 × □) + 3?

(2) **Here is a graphical picture of the truth set:**

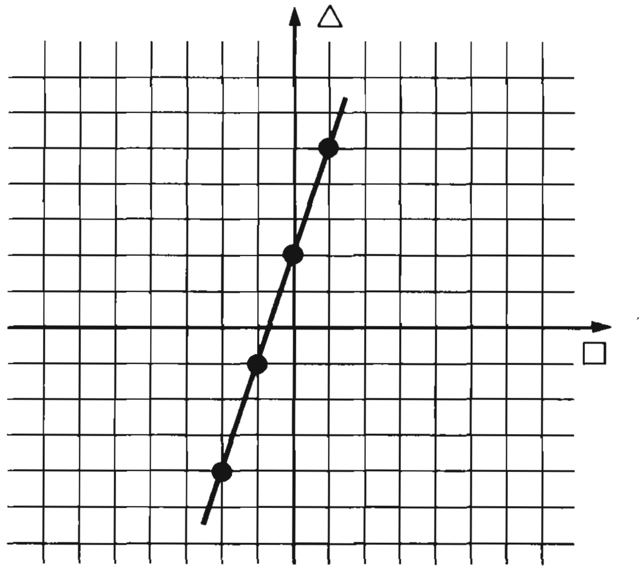


For problems 3 through 12, "Without doing the arithmetic," of course, means in reality by recognizing and using the ideas of intercept *and* slope.

Can you make a graph for each truth set **without** doing the arithmetic?

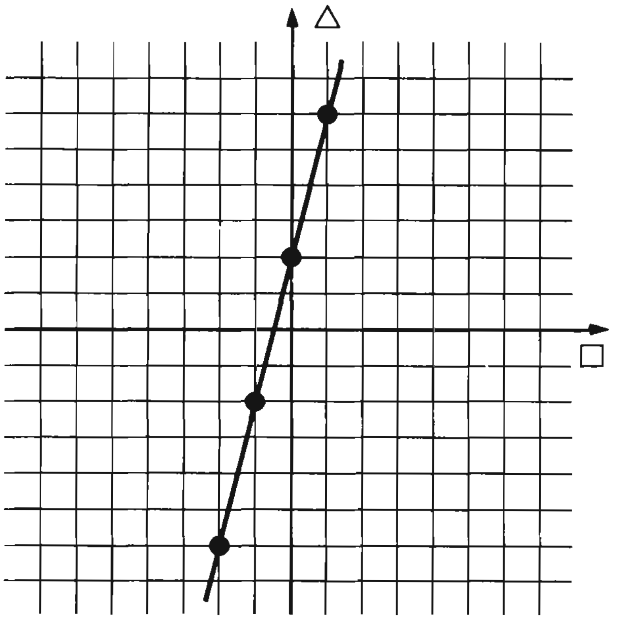
(3) $\triangle = (3 \times \square) + 2$

(3)



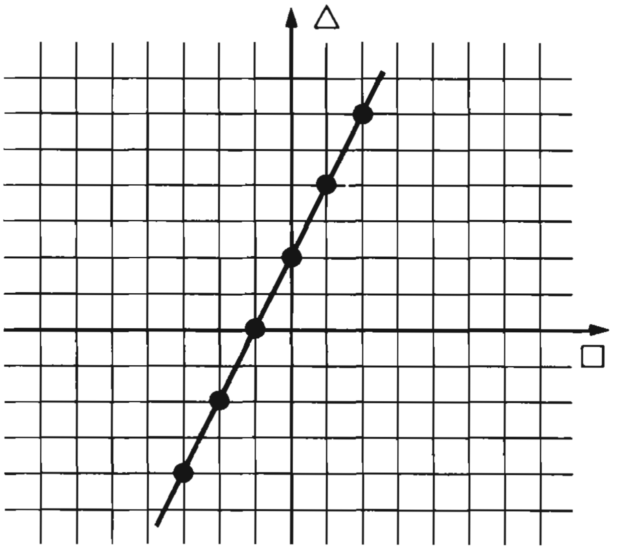
(4) $\triangle = (4 \times \square) + 2$

(4)



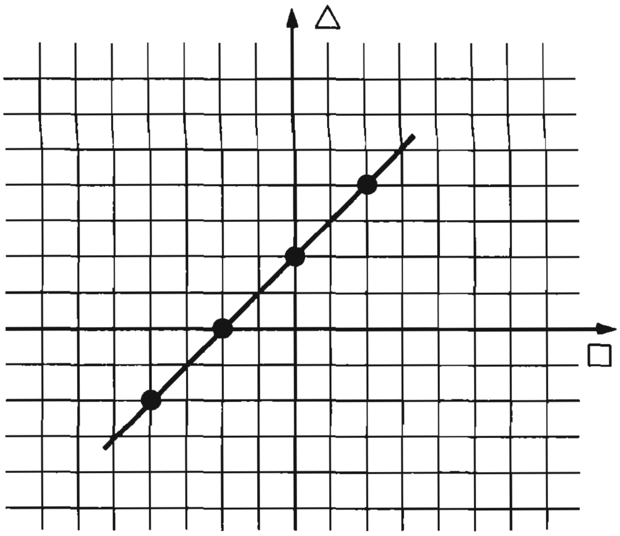
(5) $\triangle = (2 \times \square) + 2$

(5)



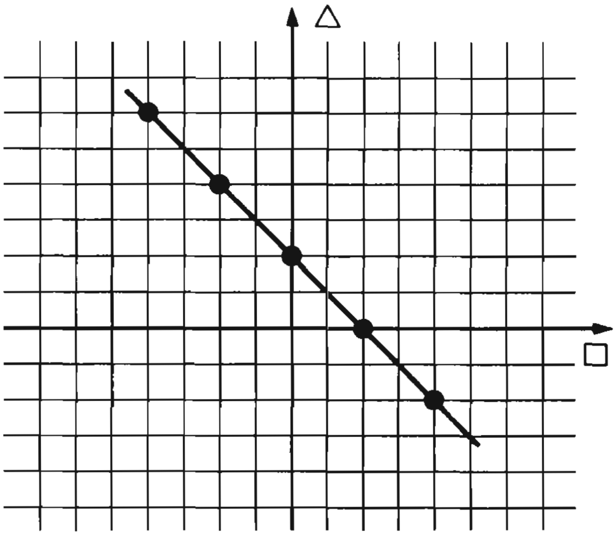
(6) $\triangle = (1 \times \square) + 2$

(6)



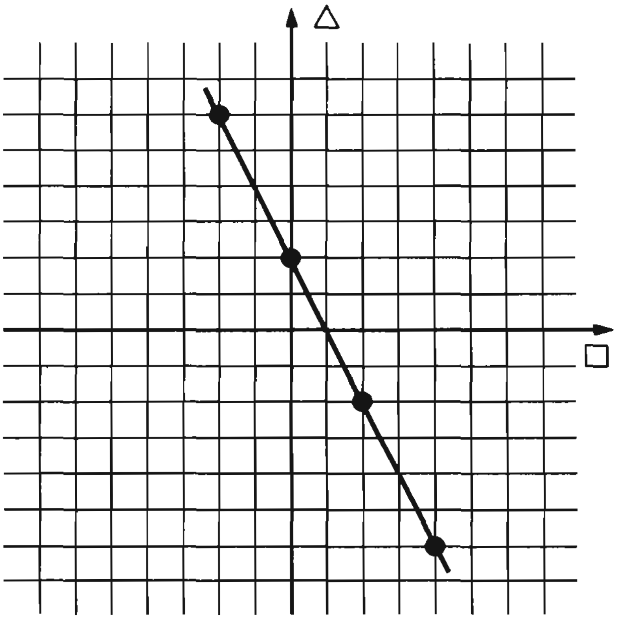
(7) $\triangle = ({}^{-}1 \times \square) + {}^{+}2$

(7)



(8) $\triangle = ({}^{-}2 \times \square) + {}^{+}2$

(8)



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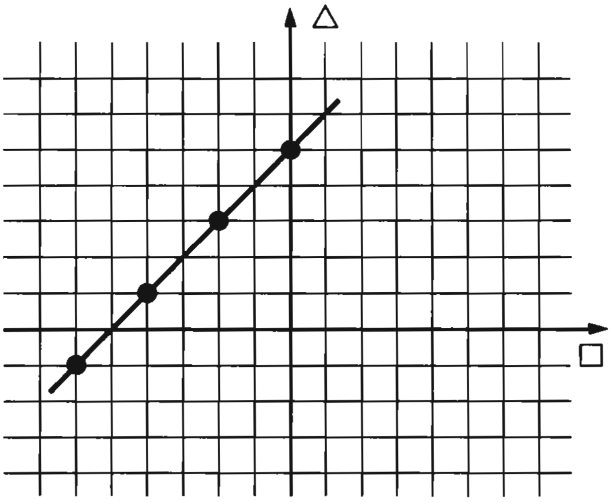
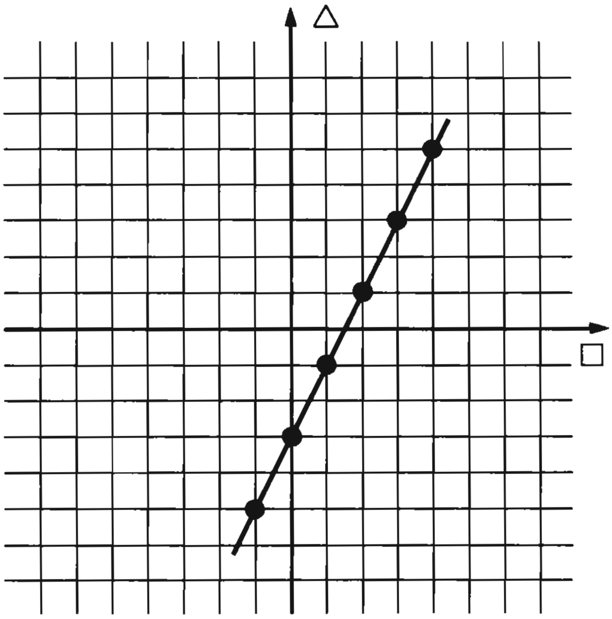
(9) $\triangle = (^{+}1 \times \square) + {}^{+}5$

(9)



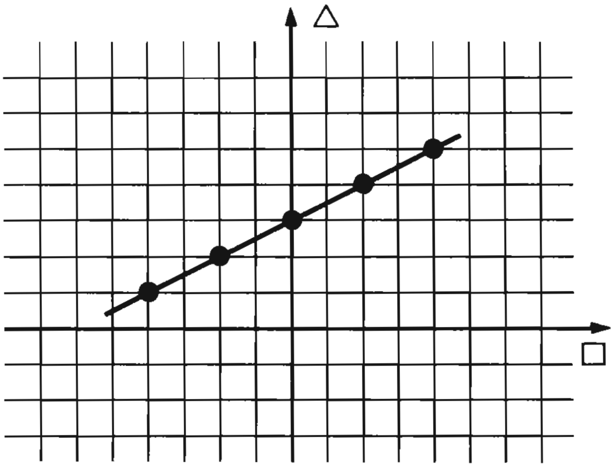
(10) $\triangle = (^{+}2 \times \square) + {}^{-}3$

(10)



(11) $\triangle = (^{+}\frac{1}{2} \times \square) + {}^{+}3$

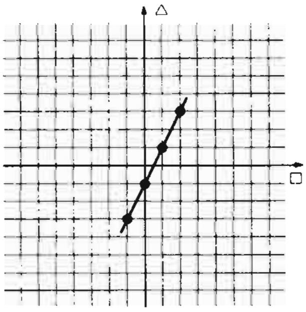
(11)



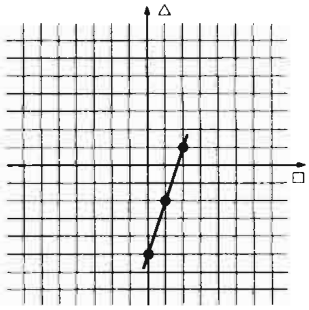
(12) $\triangle = ({}^{-}\frac{1}{2} \times \square) + {}^{+}3$

Can you write the equation for each graph?

(13)



(14)



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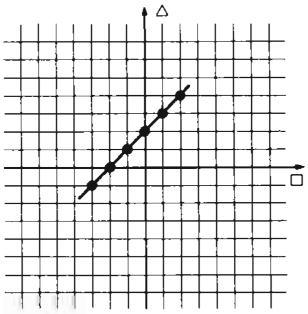

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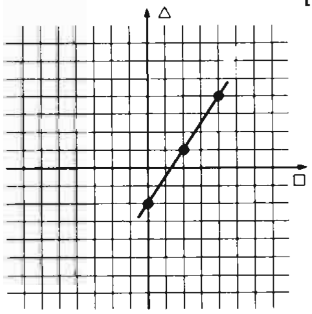
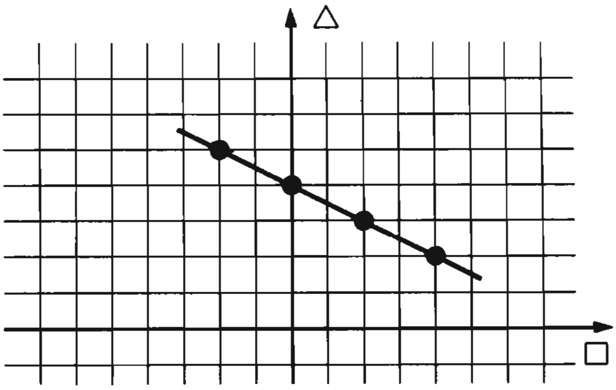


[page 70]

(12)



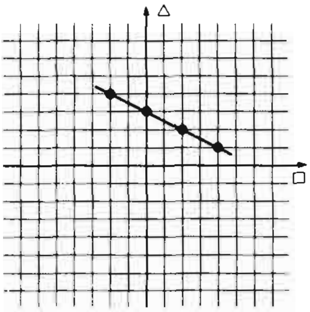
(13) $\triangle = (2 \times \square) + {}^{-}1$

(14) $\triangle = (3 \times \square) + {}^{-}5$

(15) $\triangle = \square + 2$

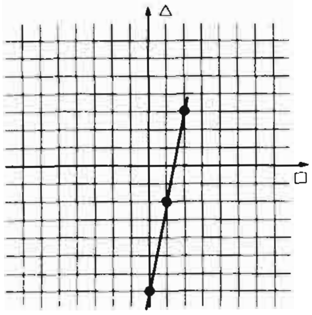
(16) $\triangle = (\frac{3}{2} \times \square) + {}^{-}2$

(17)



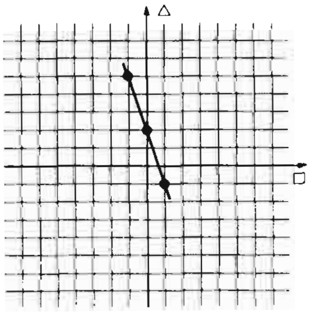
(17) $\triangle = ({}^{-}\tfrac{1}{2} \times \square) + {}^{+}3$

(18)



(18) $\triangle = (5 \times \square) + {}^{-}7$

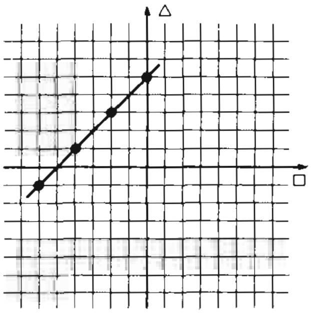
(19)



(19) $\triangle = ({}^{-}3 \times \square) + {}^{+}2$

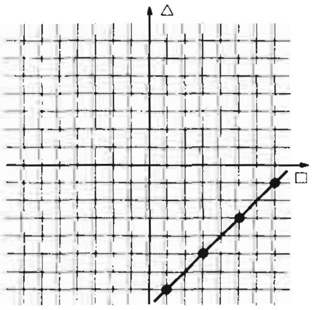
[page 71]

(20)



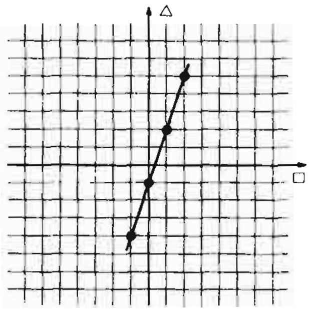
(20) $\triangle = \square + 5$

(21)



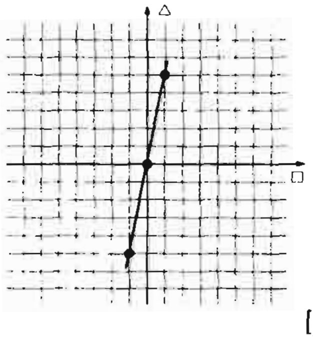
(21) $\triangle = \square + {}^{-}8$

(22)



(22) $\triangle = (3 \times \square) + {}^{-}1$

(23)



(23) $\triangle = 5 \times \square$

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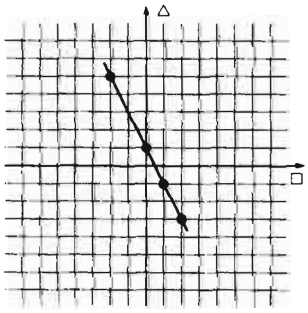
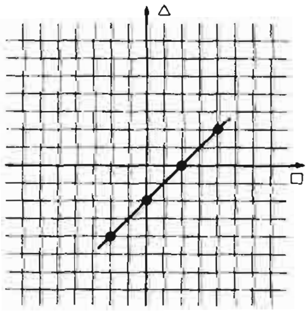
(24)



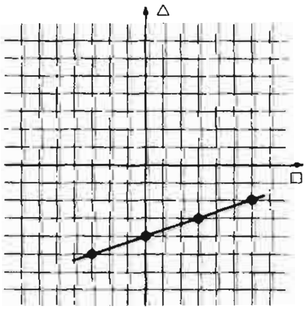
(24) $\triangle = ({}^{-}2 \times \square) + 1$

(25)



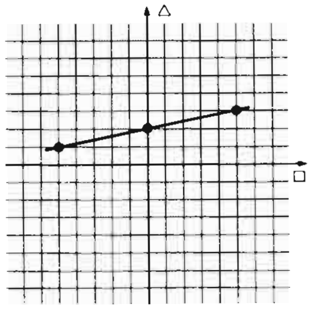
(25) $\triangle = (1 \times \square) + {}^{-}2$

(26)



(26) $\triangle = (\frac{1}{3} \times \square) + {}^{-}4$

(27)



(27) $\triangle = \left(\frac{1}{5} \times \square\right) + 2$

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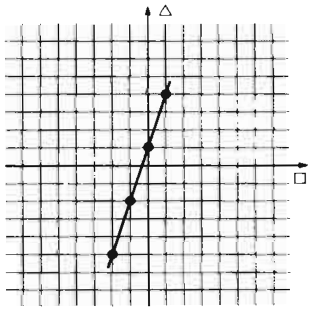
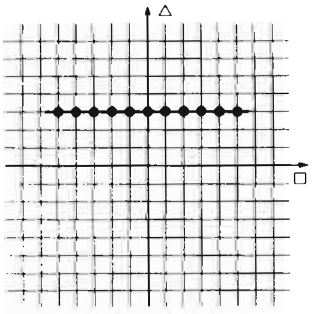
(28)



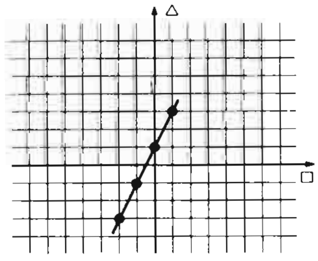
(28) $\triangle = (3 \times \square) + 1$

(29)



(29) $\triangle = 3$

(30)



(30) $\triangle = (2 \times \square) + 1$

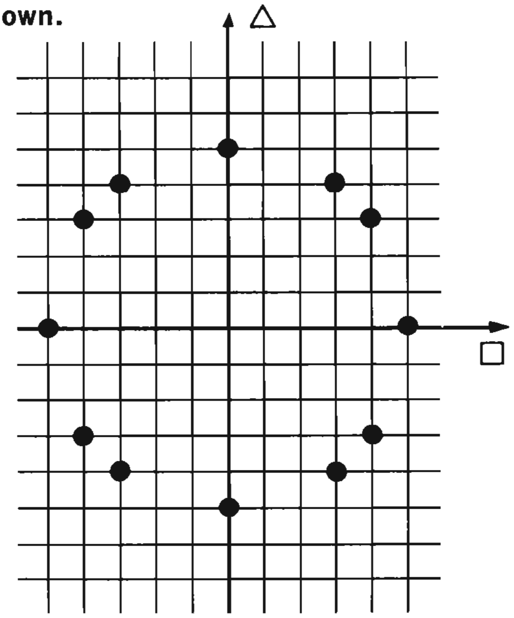
(31) Can you make a graph to show the truth set for

(□ × □) + (△ × △) = 25?

(31) Because this is something new, a *table* for part of the truth set is also shown.

□	△
3	4
3	$^-4$
$^-3$	4
4	3
$^-4$	3
4	$^-3$
$^-3$	$^-4$
$^-4$	$^-3$
5	0
0	5
$^-5$	0
0	$^-5$
⋮	⋮

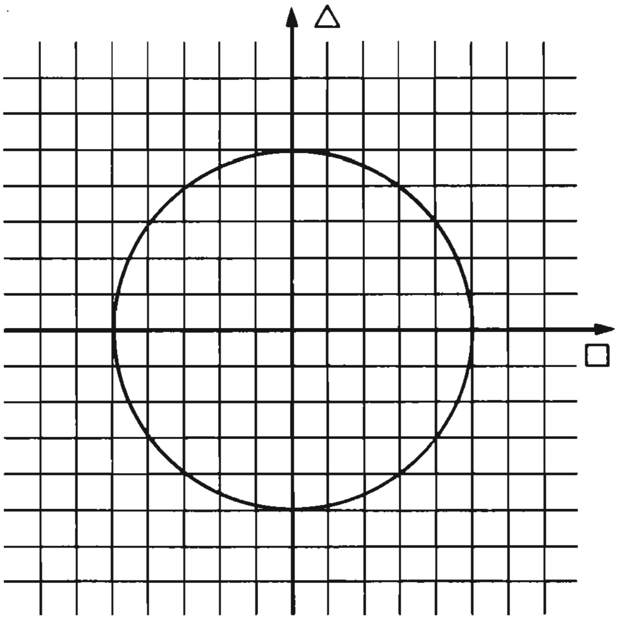
Table for truth set



Graph for truth set

For convenience in this book, the table is listed first and *then* the graph. In classroom teaching, it is probably better, for this equation, to work them out more or less simultaneously. In that way, the new number pairs help form the picture of the graph, and the gradually emerging picture of the graph suggests new number pairs.

If we now fill in the *fractional or decimal* values (actually irrational, in fact), a smooth continuous circle emerges (of radius 5, centered at the origin):



(32) How would you describe the picture in question 31?

(32) It is a circle.

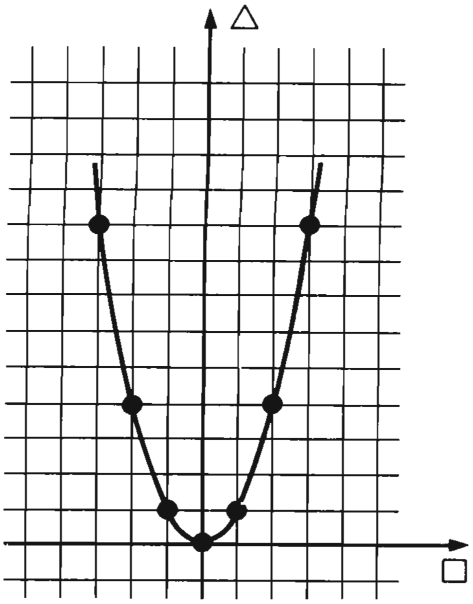
(33) See if you can make a graph to show the truth set for

△ = □ × □.

(33)

□	△
0	0
1	1
2	4
3	9
4	16
$^-1$	1
$^-2$	4
$^-3$	$^+9$
$^-4$	$^+16$
⋮	⋮

Table for truth set



Graph for truth set

(34) Do you know the name for the picture in question 33?

(34) The curve is a *parabola*.

(35) Where have you seen curves like this?

(35) Such curves appear (more or less, anyhow) in: cables of suspension bridges; radar antennas; reflectors in automobile headlights; sometimes in outside shape of spotlights or of old-fashioned auto headlights; a string of unclasped pearls held by the two ends; the trajectory of bullets or missiles or almost any thrown object.

(36) Sarah says that if you hold up a chain necklace by the two ends, it will hang in a curve that looks like a parabola. Have you ever tried it?

(36) The curve you get is (approximately) a parabola.

chapter 36 / *Pages 74–75 of Student Discussion Guide*

"THINK OF A NUMBER . . ."

This chapter is intended as preliminary preparation for word problems. The essential feature of algebraic word problems is the use of a placeholder or variable or letter or pronumeral to represent the *unknown*. That is, there is some crucial (but unknown) number that is represented by *x*, or—as in the present chapter—by a box.

Chapter 36
"THINK OF A NUMBER . . ."

[page 74]

(1) Can you find the truth set for this equation?

$$(\square \times \square) - (104 \times \square) + 305 = 2$$

ANSWERS AND COMMENTS

(1) $\{$**101, 3**$\}$

To begin with, we notice that

$$(\square \times \square) - (104 \times \square) + 305 = 2$$

is *not* in the proper form for the *two secrets* about quadratic equations. The secrets do not work properly unless the right-hand side is zero.

Well, if we *want* a zero on the right-hand side, we can get one by using a transform operation: we can *subtract 2 from each side of the equation*. This will not change the truth set.

By subtracting 2 from each side, we get:

$$(\square \times \square) - (104 \times \square) + 303 = 0.$$

We can now apply the two secrets:

$$3 \times 101 = 303,$$
$$3 + 101 = 104.$$

Therefore, the truth set is $\{101, 3\}$.

(2) Cathy: "Think of a number . . ." $\square$
Jim: "All right."
Cathy: ". . . multiply it by itself . . ." $\square \times \square$
Jim: "Uh-hum."
Cathy: ". . . take the original number and double it . . ." $2 \times \square$
Jim: "O.K."
Cathy: ". . . now subtract this
$\square \times \square$
minus this
$2 \times \square$ . . ." $(\square \times \square) - (2 \times \square)$
Jim: "All right."
Cathy: ". . . add 20 . . ."

(2) $^{+}$**7**

The first task is to write the equation representing Cathy and Jim's conversation, namely:

$$(\square \times \square) - (2 \times \square) + 20 = 55.$$

The task *now* is to put this equation into the proper form, that is, with a zero on the right-hand side (otherwise, the two secrets will not work).

Subtract 55 from each side, to get:

$$(\square \times \square) - (2 \times \square) - 35 = 0.$$

Jim: "Right."
Cathy: "... and tell me your answer."
Jim: "55."

Cathy now told Jim what number he started with. What was it?

(3) Geoff: "Think of a number ..." $\square$
Ellen: "All right."
Geoff: "... multiply it by itself ..." $\square \times \square$
Ellen: "Right."
Geoff: "... take the original number and double it ..." $2 \times \square$
Ellen: "All right."
Geoff: "... now subtract this
$\square \times \square$
minus this
$2 \times \square$..." $(\square \times \square) - (2 \times \square)$
Ellen: "All right."
Geoff: "... add 10 ..."
Ellen: "Uh-hum."
Geoff: "... and tell me the answer ..."
Ellen: "58."
Geoff: "... the number you thought of is ______."
Ellen: "That's right!"

What number did Geoff tell Ellen?

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(4) Joe: "Think of a number ..." $\square$
Daria: "All right."
Joe: "... multiply it by itself ..." $\square \times \square$
Daria: "All right."
Joe: "... take your original number and double it ..." $2 \times \square$
Daria: "O.K."
Joe: "... now subtract this
$\square \times \square$
minus this
$2 \times \square$..." $(\square \times \square) - (2 \times \square)$
Daria: "All right, I've done that."
Joe: "... add 3 ..."
Daria: "Uh-hum."
Joe: "... and tell me the answer ..."
Daria: "11."
Joe: "... all right, the number you picked was ______."
Daria: "No! Wrong!"

What number did Daria pick?

(5) Have you played this game with your friends?

We might better rewrite this as:

$$(\square \times \square) - (2 \times \square) + {}^{-}35 = 0.$$

The two secrets will now work, and we must find two numbers whose *product* is $^{-}35$, and whose *sum* is $^{+}2$.

After a little thought we see that the two numbers are $^{+}7$, $^{-}5$ and the truth set is $\{^{+}7, {}^{-}5\}$.

This is as far as we can go with *mathematics*. Mathematics tells us that Jim chose *either* $^{+}7$ or $^{-}5$.

To go further we must use psychology on Jim; we look at him carefully and ask ourselves: Jim, old fellow, did you choose $^{+}7$, or did you choose $^{-}5$. In fact, Jim chose $^{+}7$.

(3) $^{+}8$

Equation: $(\square \times \square) - (2 \times \square) + 10 = 58.$

Using transform operations (subtracting 58 from each side), we get: $(\square \times \square) - (2 \times \square) + {}^{-}48 = 0.$ We might list the factorizations of $^{-}48$: $^{-}48$, $^{+}1$; $^{+}2$, $^{-}24$; $^{+}3$, $^{-}16$; $^{+}4$, $^{-}12$; $^{+}6$, $^{-}8$; $^{+}8$, $^{-}6$; $^{+}12$, $^{-}4$; $^{+}16$, $^{-}3$; $^{+}24$, $^{-}2$; $^{+}48$, $^{-}1$. There should be *exactly one* factorization among those listed for which the *sum* is 2.

$$\left[\text{Reason: } (\square \times \square) - (\underset{\substack{\uparrow\\ \text{sum of roots}}}{2} \times \square) + {}^{-}48 = 0.\right]$$

Looking down the list, we find $^{+}8 + {}^{-}6 = 2$. Hence, the truth set must be $\{^{+}8, {}^{-}6\}$.

Either Ellen thought of $^{+}8$ or she thought of $^{-}6$, to start with. A good guess is that she started with $^{+}8$.

(4) $^{-}2$

Here is the equation: $(\square \times \square) - (2 \times \square) + 3 = 11.$ Subtracting 11 from each side, we get

$$(\square \times \square) - (2 \times \square) + {}^{-}8 = 0.$$

For this quadratic equation, these are the factorizations of $^{-}8$: $^{+}1$, $^{-}8$; $^{+}2$, $^{-}4$; $^{+}4$, $^{-}2$; $^{+}8$, $^{-}1$. The unique pair that adds to a sum $^{+}2$ is: $^{+}4$, $^{-}2$.

A good guess would ordinarily be $^{+}4$, but Daria is a very clever girl. Daria (being tricky) chose $^{-}2$.

(5) You may want to try this game in your own classroom.

chapter 37 / *Pages 75–77 of Student Discussion Guide*

MACHINES

This chapter is intended to develop (in preliminary form) the following:

(a) The concept of *general form* of an equation with literal coefficients (e.g., $ax^2 + bx + c = 0$ is the general form of a quadratic equation). Most children in traditional curricula never seem to master this notion fully.

(b) The technique of using transform operations mentally. (This is the currently fashionable counterpart of the old-fashioned notions of transposing, canceling, etc.)

In the teaching of this material (interspersed throughout five or six lessons), this method is never (or rarely) discussed. The children merely "guess" answers and are told whether they are right or wrong.

What we are here calling "machines" would, of course, be called *formulas* in usual language.*

The word *machine* was chosen to emphasize that this machine gives the (right) answer to a very wide class of problems. For example, for the equation

$$(a \times \Box) + (b \times \Box) = c + d$$

we get the machine

$$\Box = \frac{c + d}{a + b}.$$

(Use the commutative law for multiplication and the distributive law on the left-hand side; then solve for the box by dividing both sides of the equation by $a + b$.)

This machine will solve all the following equations (*of this type*, naturally!).

$$(7 \times \Box) + (1 \times \Box) = 42 + 6 \qquad \Box = \frac{42 + 6}{7 + 1} = 6$$

$$(6 \times \Box) + (1 \times \Box) = 41 + 8 \qquad \Box = \frac{41 + 8}{6 + 1} = 7$$

$$(1 \times \Box) + (1 \times \Box) = 67 + 3$$

$$\Box = \frac{67 + 3}{1 + 1} = \frac{70}{2} = 35$$

$$({}^{+}8 \times \Box) + ({}^{-}1 \times \Box) = {}^{+}12 + {}^{+}2$$

$$\Box = \frac{{}^{+}12 + {}^{+}2}{{}^{+}8 + {}^{-}1} = \frac{14}{7} = 2$$

$$(3 \times \Box) + (0 \times \Box) = 6 + 4 \qquad \Box = \frac{6 + 4}{3 + 0} = \frac{10}{3} = 3\tfrac{1}{3}$$

A tape recording or sound film of this activity is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

A bit later this is built on to get into machines in geometry—specifically, machines for area and perimeter (for rectangles, parallelograms, trapezoids, etc.).

Chapter 37
MACHINES

[page 75]

Can you find the truth set for these open sentences? 75]

(1) $3 \times \square = 18$

(2) $2 \times \square = 8$

(3) $5 \times \square = 125$

(4) Jerry made up a **machine**. For **any** equation like this

$$a \times \square = b,$$

Jerry made up a machine like this

$$\square = \_\_\_\_.$$

What was Jerry's machine?

(5) What does Jerry's machine do?

(6) Make up an equation of your own:

$$\_\_ \times \square = \_\_.$$

Does Jerry's machine work for your equation?

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(7) See if you can make up a machine that will give the truth set for

$$\square + a = b.$$

ANSWERS AND COMMENTS

(1) **{6}**

(2) **{4}**

(3) **{25}**

(4) $\square = \frac{b}{a}$

(5) **It indicates the truth set for *all* equations of the type**

$$a \times \square = b,$$

no matter what numbers* were put into the places marked *a* and *b*.

(6) **Presumably it does.**

It is well to bear in mind that Jerry's machine *fails* if and only if the number *zero* is used in the place marked *a*. Most children would discover this exceptional case for themselves *right at this stage* if they were asked the *right* question, for example: "Can you find *any* number to put in the places *a* and *b*, *so that Jerry's machine won't work for your number?*" However, it seems preferable to wait before raising this issue.

In any event, the purpose is to help children to formulate the essential concepts of mathematics—not to write a treatise on logic which must be defended against every mathematical Kentucky lawyer. (One approaches these two tasks somewhat differently.)

(7) $\square = b - a$

---

* Well, this is not *exactly* true, but none of the children has ever cited the exceptional case *at this point in the course*. They do later on, and later on, therefore, we deal with this *exceptional* case.

Incidentally, for some reason machines seem to evoke more *guessing* by the children than many adults ordinarily expect. The children approach machines with *gleeful* and *random* guesses. This is not objectionable, and, indeed, a careful "system" gradually emerges as this topic is returned to intermittently in subsequent lessons. A discreet intermingling of *balance pictures* in later lessons helps to move from the stage of random guesses to the stage of insightful method.

(8) Make up an equation of your own with numbers for a and b:

$$\square + \_\_ = \_\_.$$

Does the machine from question 7 work for your own equation?

(8) **Presumably it does. This time there are no exceptional cases.**

(9) Can you make up a machine that will give the truth set for

$$\square + a + b = c + d?$$

(9) **There are many ways to write a suitable machine for this problem. Here are a few:**

$$\square = (c + d) - (a + b)$$

$$\square = (c + d) - (b + a)$$

$$\square = (d + c) - (b + a)$$

$$\square = c + d - b - a$$

This, skillfully developed at the right moment (perhaps not just yet), can be very provocative, and lead to many useful discoveries, such as the discovery of the identity

$$(a + b) - (c + d) = a + b - c - d.$$

(10) Make up an equation of your own with numbers for a, b, c, and d. Does the machine of question 9 work for your equation?

(10) **Presumably. Try it and see.**

Try to make up a machine to give the truth set for each equation.

(11) $\square + a + b + c = w + x + y$

(11) **Here are a few correct answers:**

$$\square = (w + x + y) - (a + b + c)$$

$$\square = (y + x + w) - (c + b + a)$$

$$\square = y + x + w - c - b - a$$

(12) $(a \times \square) + b = c$

(12) $$\square = \frac{c - b}{a}$$

(13) $(a \times \square) + b + c = r + s$

(13) $$\square = \frac{(r + s) - (b + c)}{a}$$

or

$$\square = \frac{r + s - b - c}{a}$$

etc.

(14) $(a \times \square) + b + c + d = r + s + u$

(14) Here are a few correct answers:

$$\square = \frac{(r + s + u) - (b + c + d)}{a}$$

$$\square = \frac{(u + s + r) - (d + c + b)}{a}$$

$$\square = \frac{r + s + u - b - c - d}{a}$$

(15) $(a \times \square) + b + c - d = r + s - u$

(15) $$\square = \frac{(r + s + u) - (b + c - d)}{a}$$

or

$$\square = \frac{r + s + u - b - c + d}{a}$$

etc.

(16) $(a \times \square) + b + c + d = r + s - u$

(16) $$\square = \frac{(r + s - u) - (b + c + d)}{a}$$

or

$$\square = \frac{r + s - u - b - c - d}{a}$$

etc.

(17) $(a \times \square) + b + c - d = r + s - u$

(17) $$\square = \frac{(r + s - u) - (b + c - d)}{a}$$

or

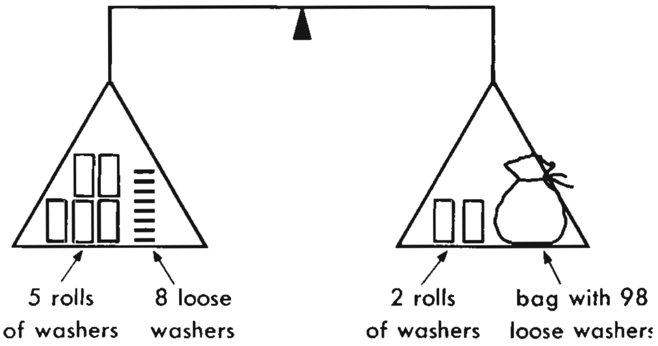
$$\square = \frac{r + s - u - b - c + d}{a}$$

etc.

(18) See if you can make up a balance picture for this equation.

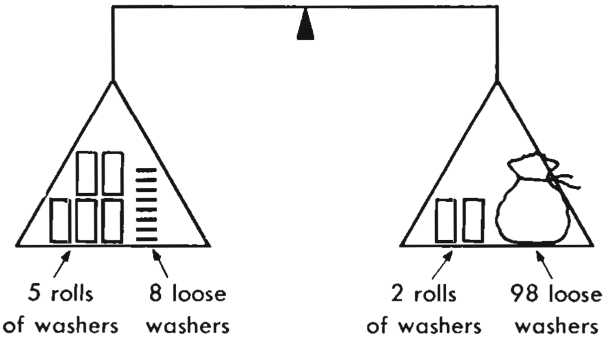
$(5 \times \square) + 8 = (2 \times \square) + 98$

(18)



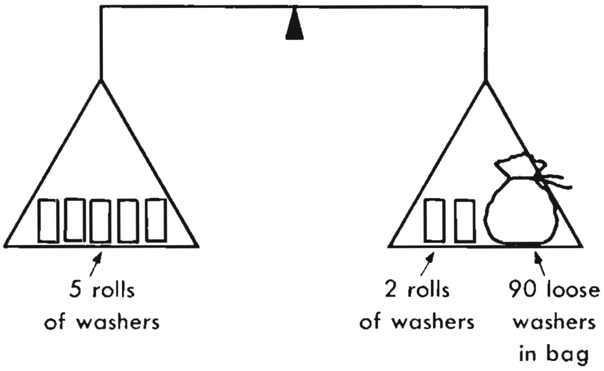
(19) Pretend that you are very stupid. Use *transform operations* to simplify the equation in question 18. Can you make a balance picture at each step?

(19) There are, of course, many ways to do this. Here is one sequence of step-by-step simplifications:



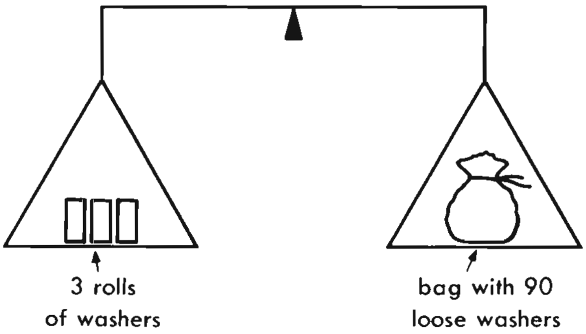
$$(5 \times \square) + 8 = (2 \times \square) + 98$$

Remove eight loose washers from each balance pan:



$$(5 \times \square) = (2 \times \square) + 90$$

Remove two rolls from each side:



$$(3 \times \square) = 90$$

This last equation is probably simple enough that we can solve it, and see that the truth set must be $\{30\}$.

Since, at each step in this simplification a legitimate *transform operation* was used, the truth set was not changed at any step. Hence {30} must also be the truth set for the original equation, $(5 \times \square) + 8 = (2 \times \square) + 98$. Of course, many of your students can merely look at

$$(5 \times \square) + 8 = (2 \times \square) + 98$$

and see immediately that the truth set is {30}.

(We are pretending to be very clever at transform operations, but very stupid at finding truth sets.)

(20) What does a machine do?

(20) A machine (or formula) solves every equation of a certain type.

For example, the machine $\square = b - a$ solves every equation of the type $\square + a = b$.

It solves (i.e., simplifies) every equation such as:

Equation	Substitution	Result
$\square + 7 = 13$	$b = 13$	$\square = 13 - 7$
	$a = 7$	$\square = 6$
$\square + 5 = 301$	$b = 301$	$\square = 301 - 5$
	$a = 5$	$\square = 296$
$\square + {}^{-}3 = {}^{-}18$	$b = {}^{-}18$	$\square = {}^{-}18 - {}^{-}3$
	$a = {}^{-}3$	$\square = {}^{-}15$

The idea is that the *machine* equation, for example, $\square = 6$, shows its truth set (i.e., {6}) more clearly than the original equation did (i.e., $\square + 7 = 13$).

(21) Do you know the word that mathematicians use to refer to machines?

(21) Formulas

Can you make up a machine to find the truth set for each equation?

(22) $\square + a - b + c = r + s - u$

(22) $\square = (r + s - u) - (a - b + c)$

or

$\square = r + s - u - a + b - c$

etc.

(23) $(\square \times a) + (\square \times b) = c$

(23) $\square = \dfrac{c}{a + b}$ **or** $\square = \dfrac{c}{b + a}$

Problem 23 is somewhat tricky. Try asking the children to *use an identity to rewrite* $(\square \times a) + (\square \times b) = c$ *so as to make it easier to find a machine.* The answer is to use the distributive law:

$$A \times (B + C) = (A \times B) + (A \times C),$$

putting $\square \rightarrow A$, $a \rightarrow B$, $b \rightarrow C$, so as to get

$$\square \times (a + b) = (\square \times a) + (\square \times b). \qquad (1)$$

Now, equation (1) says that

$$(\square \times a) + (\square \times b) \qquad (2)$$

and

$$\square \times (a + b) \qquad (3)$$

are two different names for the same thing. (This is the meaning of the equals sign.)

In $(\square \times a) + (\square \times b) = c$ we have the *first* of these two names [equation (2)]. Thus, we may replace equation (1) by the *second* name for the same thing, i.e., $\square \times (a + b)$. If we do so, we get $\square \times (a + b) = c$.

For this equation, it is clear that a machine would be:

$$\square = \frac{c}{a + b}.$$

(24) $(\square \times r) + (\square \times s) = v$

(24) $\square = \frac{v}{r + s}$. **See answer to 23.**

[page 77]

(25) $(\square \times a) + (\square \times 7) = c$

(25) $\square = \frac{c}{a + 7}$

(26) $(\square \times a) + \square = c$

(26) $\square = \frac{c}{a + 1}$

This is a hard problem. Fourth or fifth graders do not usually get it immediately. One way to handle it is to *use identities* to rewrite the equation $(\square \times a) + \square = c$ before trying to find a machine. The identities to use are:

$\square \times 1 = \square$ The law for one

$\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$

The distributive law

Here is how:

$$\begin{aligned} (\square \times a) + \square &= c \\ &\qquad \text{L1} \\ (\square \times a) + (\square \times 1) &= c \\ &\qquad \text{DL} \\ \square \times (a + 1) &= c \end{aligned}$$

Now, the machine is

$$\square = \frac{c}{a + 1}.$$

(27) Jimmy made this machine for question 26.

$$\square = \frac{c}{a + \square}$$

Try Jimmy's machine on some equations with numbers for a and c. Does Jimmy's machine work?

(27) **No**

Let's see why Jimmy's machine does not work. Here is an equation we can try: $(\square \times 7) + \square = 24$.

Jimmy's machine says,

$$\square = \frac{24}{7 + \square}.$$

This does not give us any idea of what number to put into the box in order to get a *true* statement.

A good machine must always be of the form,

$$\square = \text{(something or other)},$$

where there are no boxes on the right-hand side of the equation.

As a matter of fact, the equation $(\square \times 7) + \square = 24$ can be rewritten:

$$(\square \times 7) + (\square \times 1) = 24$$

$$(\square \times 8) = 24,$$

and the truth set is, clearly, $\{3\}$.

If we put "3" into the box in Jimmy's machine, we find that

$$\boxed{3} = \frac{24}{7 + \boxed{3}}$$

or

$$3 = \frac{24}{10},$$

which is actually *false*.

Since Jimmy's machine has boxes *on both sides of the equals sign*, it cannot be a useful machine.

Since the statement

$$\boxed{3} = \frac{24}{7 + \boxed{3}}$$

is *false*, we know also that Jimmy's machine *does not even indicate the correct truth set.* Either flaw alone would be fatal. Jimmy's machine is a sad error.

(28) Al says he thinks we need to use an **identity** to rewrite the equation

$$(\square \times a) + \square = c,$$

before we try to make up a machine to give the truth set.

Which parts of the equation would you like to keep the same? Which parts would you like to change?

(28) **See question 29.**

(29) Tom made this picture to show which parts of the equation he would like to change.

$$(\square \times a) + \square \quad = \quad c$$

What do you think?

(29) **Tom's picture is right.**

(30) What **identities** should we use?

(30) **The law of one and the distributive law.**

(31) How can we rewrite the equation?

(31)

$$\begin{aligned} (\square \times a) + \square &= c \\ (\square \times a) + (\square \times 1) &= c \quad \textbf{L1} \\ \square \times (a + 1) &= c \quad \textbf{DL} \end{aligned}$$

Note: Compare questions 28 through 31 with the answer to question 26.

(32) Can you make a machine for

$$(\square \times a) + \square = c?$$

(32) $\square = \frac{c}{a + 1}$ **(See answer to question 31 above.)**

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MORE MACHINES

Chapter 38
MORE MACHINES

[page 78]

Can you make up a machine for each equation?

(1) $\square + a = b$

(2) $a \times \square = b$

(3) $\square - a = b$

(4) $\dfrac{\square}{a} = b$

(5) $\square + a + b = r$

(6) $\square + a + b + c = r + s + u$

(7) $\square + a - b + c = r + s + u$

(8) $\square + a - (b + c) = r + s + u$

ANSWERS AND COMMENTS

(1) $\square = b - a$

(2) $\square = \dfrac{b}{a}$

(3) $\square = b + a$

(4) $\square = b \times a$

(5) $\square = r - (a + b)$

or

$\square = r - (b + a)$

or

$\square = r - a - b$

or

$\square = r - b - a$

(6) $\square = (r + s + u) - (a + b + c)$

or

$\square = r + s + u - a - b - c$

etc.

(7) $\square = (r + s + u) - (a - b + c)$

or

$\square = r + s + u - a + b - c$

etc.

(8) $\square = [r + s + u] - [a - (b + c)]$

or

$\square = r + s + u - a + (b + c)$

or

$$\square = (r + s + v) - a + (b + c)$$

etc.

(9) $(a \times \square) + b = c$

(9) $\square = \dfrac{c - b}{a}$

(10) $(a \times \square) + b + c = w + h$

(10) $\square = \dfrac{(w + h) - (b + c)}{a}$

or

$$\square = \frac{w + h - b - c}{a}$$

etc.

(11) $(\square \times a) + (\square \times b) = c$

(11) $\square = \dfrac{c}{a + b}$ **(This is our old friend that can be simplified by using the distributive law.)**

(12) $(\square \times a) + (\square \times 3) = w$

(12) $\square = \dfrac{w}{a + 3}$ **(Another equation that may call for the distributive law *before* you try to find a "machine.")**

(13) $(\square \times r) + (\square \times s) = w$

(13) $\square = \dfrac{w}{r + s}$ **(See problems 11 and 12 immediately preceding.)**

(14) $(\square \times a) + \square = c$

(14) $\square = \dfrac{c}{a + 1}$ **(See Chapter 37.)**

(15) $(a \times \square) + b = (c \times \square) + w$

(15) Here is a machine:

$$\square = \frac{w - b}{a - c}$$

Problem 15 is tricky. If the children are allowed to guess, sooner or later (especially with bright children) some of them get a good enough insight to be able to guess correctly. At this stage they are not really *guessing*, but they usually claim that they are just guessing.

Alternatively (this is one of the choices that depends upon the teacher's intuitive judgment of the needs of her children), you might proceed this way:

$$(a \times \square) + b = (c \times \square) + w.$$

Subtract b from both sides (a "legal" transform operation that will not change the truth set):

$$(a \times \square) = (c \times \square) + w - b.$$

Subtract $c \times \square$ from each side:

$$(a \times \square) - (c \times \square) = w - b.$$

Use the commutative law for multiplication twice, to get:

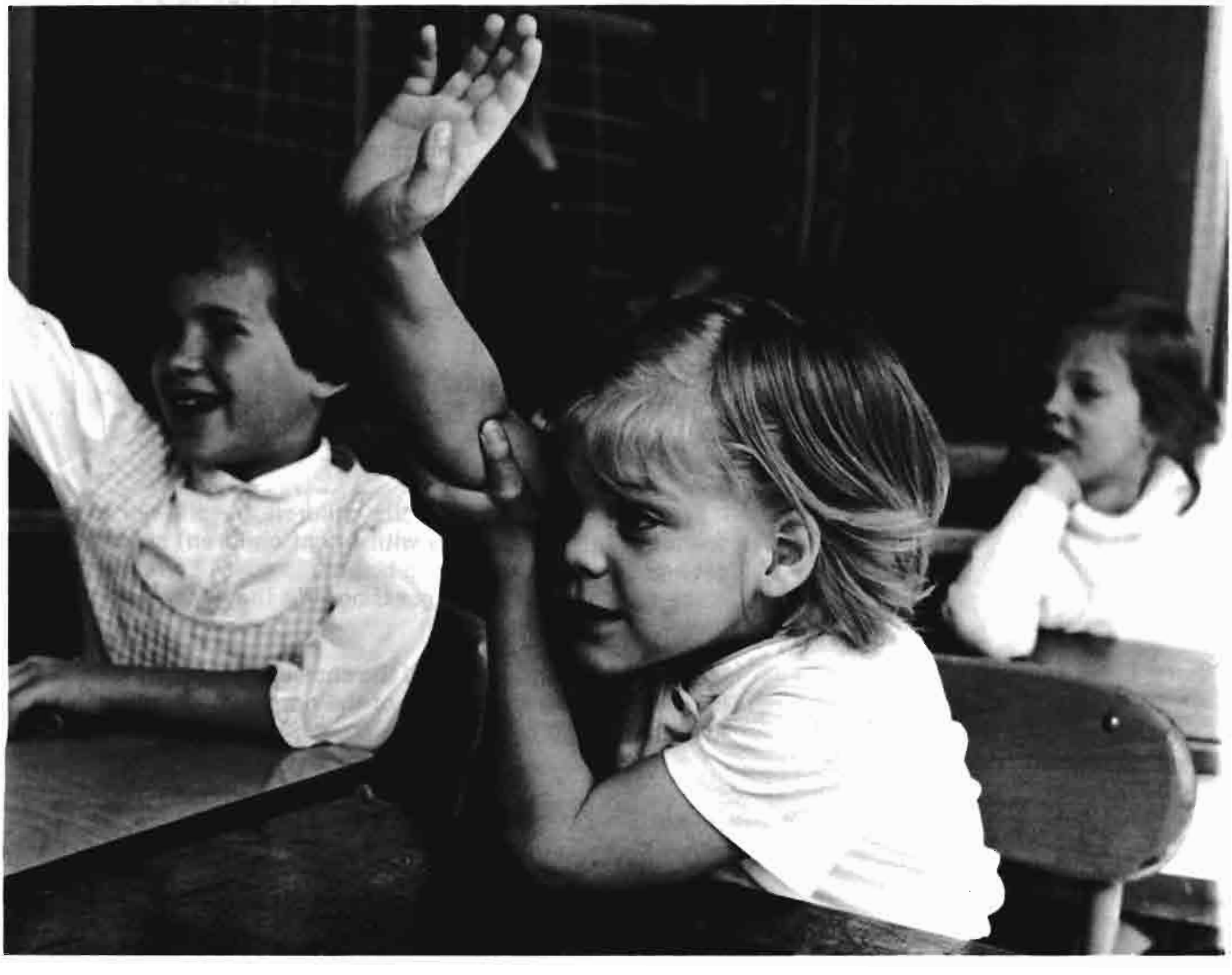
$$(\square \times a) - (\square \times c) = w - b.$$

For the next step, if you have not yet developed identities involving subtraction, you can resort to guessing by the children as to what they think it should *probably* be. The *right* answer is

$$\square \times (a - c) = w - b.$$

Consequently, the proper machine is:

$$\square = \frac{w - b}{a - c}.$$



chapter 39 / ***Page 79 of Student Discussion Guide***

THE ASSOCIATIVE LAWS

In your class you *may* have introduced the associative laws prior to this lesson. If you have not already introduced them, you can make use of the generalized associative property for addition without *calling this to the children's attention.* This is one case where what they don't know (or haven't yet thought of) won't hurt them (for the time being). Here is a method for using the generalized associative property prior to introducing the associative laws.

Actually, an expression such as

$$(A \times C) + (A \times D) + (B \times C) + (B \times D)$$

is meaningless unless we say which additions are to be performed first. (This is a consequence of the fact that *addition is a binary operation*—you always add *two* numbers at each step.)

We can put in brackets and parentheses like this:

$$[(A \times C) + (A \times D)] + [(B \times C) + (B \times D)],$$

or in some other way. This eliminates all ambiguities, and makes everything strictly legal. Propriety, however, is not without its drawbacks. If we carefully insert all the brackets and parentheses at every step, the derivation of the theorem

$$\begin{aligned}(A + B) \times (C + D) = (A \times C) &+ (A \times D)\\ &+ (B \times C) + (B \times D)\end{aligned}$$

becomes quite complicated.

Since children never seem to notice, at first, that addition is a binary operation, you can avoid telling them about this until you are good and ready.

At first, let the children assume that there is a universal associative law that says:

$$\begin{aligned}A + B + C + D + E &= (A + B) + (C + D) + E\\ &= [(A + B) + (C + D)] + E\\ &= (A + B) + [(C + D) + E]\\ &= A + (B + C) + (D + E)\\ &= [A + (B + C)] + (D + E)\\ &= A + [(B + C) + (D + E)]\\ &= (A + B) + C + (D + E)\\ &\qquad\text{etc.}\end{aligned}$$

They never mention this law, and you need not. You and the children can just accept it without comment or question. This is something like believing in Santa Claus. Properly used, it can be a *good thing* . . . .

But . . . we are about to sample the hitherto untasted fruit of *knowledge*, (or wisdom or scepticism or doubt or learning or sophistication or whatever it is).

Here are two new identities:

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

The associative law for addition (ALA)

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$

The associative law for multiplication (ALM)

Chapter 39
THE ASSOCIATIVE LAWS

ANSWERS AND COMMENTS

[page 79]	
(1) What do we mean by an **identity?**	(1) **An open sentence that becomes true for every correctly made substitution.**
(2) How many identities are there?	(2) **A very, very, very long list . . . in fact, it never stops. You can keep making up new identities as long as you want.**
(3) What do we mean by an **axiom?** What is special about axioms?	(3) **If we shorten this huge long list of identities, and get the *shortest possible list* from which we can derive all of the other identities of algebra, then: the statements that *remain* on our shortest possible list are called *axioms*, and the statements that can be eliminated because they are not really necessary are called *theorems*.** **The *axioms* imply the *theorems*. That is why we can delete the *theorems* without really losing anything. To show this implication in detail, we make up a *derivation*.**
(4) What do we mean by a **theorem?**	(4) **See the answer to question 3.**
(5) Is $A + (B \times C) = (C \times B) + A$ an axiom or a theroem? Why?	(5) **It is a *theorem*, because we can derive it from the commutative laws for multiplication and addition.**
(6) See if you can write down the list of axioms.	(6) **Because of our two new identities, this list has just been lengthened. It now reads (subject to minor variation between one class and another):** $\square = \square$ $\square + \triangle = \triangle + \square$ $\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$ $\square \times 1 = \square$ $\square \times \triangle = \triangle \times \square$ $\frac{\square}{\square} = 1$

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

Associative law for addition

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$$

Associative law for multiplication

$\square \times 0 = 0$ **(Actually a theorem, but we usually prefer to list it as an axiom.)**

$$\square + 0 = \square$$

$$\left.\begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ 4 + 1 = 5 \\ \quad\vdots \end{array}\right\} \text{Changing names*}$$

(7) Did you include

$$A + (B \times C) = (C \times B) + A$$

on your list? Why?

(7) No. It is a *theorem*, not an *axiom*. Recall the derivation:

$$\begin{array}{ll} A + (B \times C) = A + (B \times C) & \\ & \text{CLA} \\ A + (B \times C) = (B \times C) + A & \\ & \text{CLM} \\ A + (B \times C) = (C \times B) + A & \end{array}$$

Q.E.D.

(8) Jerry says he has found a new identity which should be included on the list, namely

$$\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$$

What do you think?

(8) Jerry is right.

(9) Cindy says Jerry's new identity is called the **associative law for addition**. What do you think?

(9) Cindy is also right.

(10) Debbie says she knows **another** identity that should also be added to the list of axioms. Can you guess what it is?

(10) By analogy with the associative law for addition, most children guess, correctly, that it is:

$$\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown \quad \text{ALM}$$

(11) Francis says we should add this identity

$$\square + (\triangle \times \bigtriangledown) = (\square + \triangle) \times \bigtriangledown$$

to the list of axioms.

Do you agree?

(11) No. Francis' open sentence is *not* an identity. (It becomes *false* for most substitutions.)

See if you can make a derivation for each theorem.

(12) $(A + B) + C = (C + B) + A$

(12) Theorem: $(A + B) + C = (C + B) + A$

Proof:

$$\begin{array}{ll} (A + B) + C = (A + B) + C & \\ & \text{ALA} \\ (A + B) + C = A + (B + C) & \\ & \text{CLA} \\ (A + B) + C = (B + C) + A & \\ & \text{CLA} \\ (A + B) + C = (C + B) + A & \end{array}$$

Q.E.D.

---

* These are the only axioms that are not *identities*, for the reason—obviously—that they contain no boxes. In order to include these, we should probably speak of the axioms as a list of statements rather than as a list of identities. However, the latter is more effective language for communication with the children.

(13) $(A + B) + (C + D) = (D + B) + (C + A)$

(13) **Theorem:** $(A + B) + (C + D) = (D + B) + (C + A)$

Proof:

$(A + B) + (C + D) = (A + B) + (C + D)$

CLA

$(A + B) + (C + D) = (A + B) + (D + C)$

CLA

$(A + B) + (C + D) = (B + A) + (D + C)$

ALA*

$(A + B) + (C + D) = [(B + A) + D] + C$

ALA

$(A + B) + (C + D) = [B + (A + D)] + C$

CLA

$(A + D) + (C + D) = [B + (D + A)] + C$

ALA

$(A + B) + (C + D) = [(B + D) + A] + C$

CLA

$(A + B) + (C + D) = [(D + B) + A] + C$

ALA†

$(A + B) + (C + D) = (D + B) + (A + C)$

CLA

$(A + B) + (C + D) = (D + B) + (C + A)$

Q.E.D.

(14) $2 \times 3 = 6$

(14) **Theorem:** $2 \times 3 = 6$

Proof:

$2 \times 3 = 2 \times 3$

CN

$2 \times 3 = (1 + 1) \times 3$

CLM

$2 \times 3 = 3 \times (1 + 1)$

DL

$2 \times 3 = (3 \times 1) + (3 \times 1)$

L1

$2 \times 3 = 3 + (3 \times 1)$

L1

$2 \times 3 = 3 + 3$

CN

$2 \times 3 = 3 + (2 + 1)$

CLA

$2 \times 3 = 3 + (1 + 2)$

ALA

$2 \times 3 = (3 + 1) + 2$

CN

$2 \times 3 = 4 + 2$

CN

$2 \times 3 = 4 + (1 + 1)$

ALA

$2 \times 3 = (4 + 1) + 1$

CN

$2 \times 3 = 5 + 1$

CN

$2 \times 3 = 6$

Q.E.D.

---

* In this use of the associative law for addition

$$\text{(i.e., } \square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown\text{)}$$

we have *four* letters (A, B, C, and D) to substitute into *three* places ($\square$, $\triangle$, and $\bigtriangledown$). We substitute $(B + A) \rightarrow \square$, $D \rightarrow \triangle$, and $C \rightarrow \bigtriangledown$. Write this out in full detail and you will see that it works.

† Into $\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$, we substitute

$(D + B) \rightarrow \square$

$A \rightarrow \triangle$

$C \rightarrow \bigtriangledown$.

(15) $2 \times 4 = 8$

(16) $\square + \square = 2 \times \square$

(17) $3 + 5 = 8$

(18) $4 + 4 = 8$

(15) **Similar to problem 14.**

(16) **Theorem:** $\square + \square = 2 \times \square$

Proof:

$$\begin{array}{rcl}
2 \times \square = 2 \times \square & & \\
 & \big\downarrow & \textbf{CN} \\
(1 + 1) \times \square = 2 \times \square & & \\
 & \big\downarrow & \textbf{CLM} \\
\square \times (1 + 1) = 2 \times \square & & \\
 & \big\downarrow & \textbf{DL} \\
(\square \times 1) + (\square \times 1) = 2 \times \square & & \\
 & \big\downarrow & \textbf{L1} \\
\square + (\square \times 1) = 2 \times \square & & \\
 & \big\downarrow & \textbf{L1} \\
\square + \square = 2 \times \square & &
\end{array}$$

Q.E.D.

(17) **Theorem:** $3 + 5 = 8$

Proof:

$$\begin{array}{rcl}
3 + 5 = 3 + 5 & & \\
 & \big\downarrow & \textbf{CLA} \\
3 + 5 = 5 + 3 & & \\
 & \big\downarrow & \textbf{CN} \\
3 + 5 = 5 + (2 + 1) & & \\
 & \big\downarrow & \textbf{CLA} \\
3 + 5 = 5 + (1 + 2) & & \\
 & \big\downarrow & \textbf{ALA} \\
3 + 5 = (5 + 1) + 2 & & \\
 & \big\downarrow & \textbf{CN} \\
3 + 5 = 6 + 2 & & \\
 & \big\downarrow & \textbf{CN} \\
3 + 5 = 6 + (1 + 1) & & \\
 & \big\downarrow & \textbf{ALA} \\
3 + 5 = (6 + 1) + 1 & & \\
 & \big\downarrow & \textbf{CN} \\
3 + 5 = 7 + 1 & & \\
 & \big\downarrow & \textbf{CN} \\
3 + 5 = 8 & &
\end{array}$$

Q.E.D.

(18) **Similar to problem 17.**

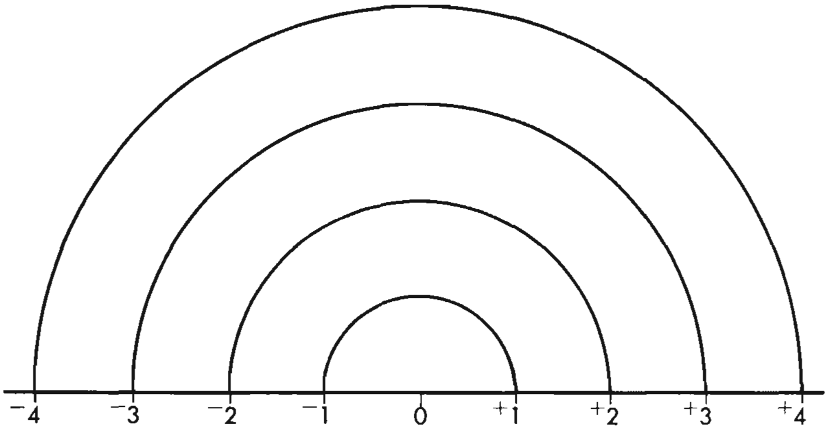
chapter 40 / *Pages 80–81 of Student Discussion Guide*

SUBTRACTION

Our list of axioms will now grow still longer, since we shall now introduce identities involving *subtraction*. We can do this easily by means of three things introduced in this chapter:

(a) the opposite of a number,
(b) the identity known as the law of opposites,
(c) the identity known as "changing signs."

The concept of *opposite* of a number (the number at the other end of the rainbow) is shown in this picture:



Thus, the *opposite* of $^{+}2$ is $^{-}2$, the *opposite* of $^{-}2$ is $^{+}2$, the *opposite* of 0 is 0, the *opposite* of $^{+}4$ is $^{-}4$, etc.

We can write the opposite of a number by using a small circle:

$$^{\circ}(^{+}2) = {}^{-}2$$
$$^{\circ}(^{-}2) = {}^{+}2$$
$$^{\circ}(0) = 0$$
$$^{\circ}(^{+}4) = {}^{-}4$$
etc.

In addition to this idea of *opposites*, we shall need two new identities:

$\square + {}^{\circ}\square = 0$ The law of opposites (L Opp)

and

$\square - \triangle = \square + {}^{\circ}\triangle$ Changing signs (CS)

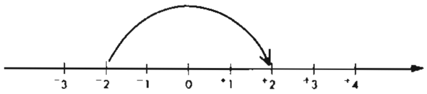
("changing signs" permits the elimination of subtraction problems by turning them into addition problems).

Chapter 40
SUBTRACTION

[page 80]

(1) Jerry says our axioms seem to be all right for *addition* and *multiplication* problems, but we don't have any axioms for *subtraction.*
What do you think?

(2) Ellen made this picture



to show that the *opposite* of $^{-}2$ is $^{+}2$,

$$^{\circ}(^{-}2) = {}^{+}2.$$

See if you can make a picture to show

$$^{\circ}(^{+}3) = {}^{-}3.$$

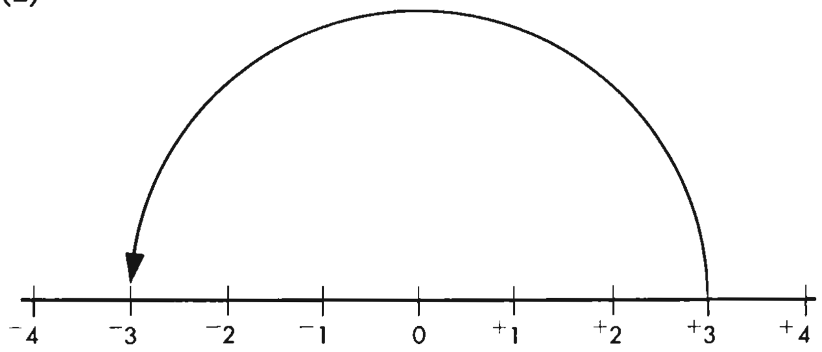
(3) See if you can make a picture to show

$$^{\circ}(^{-}5) = {}^{+}5.$$

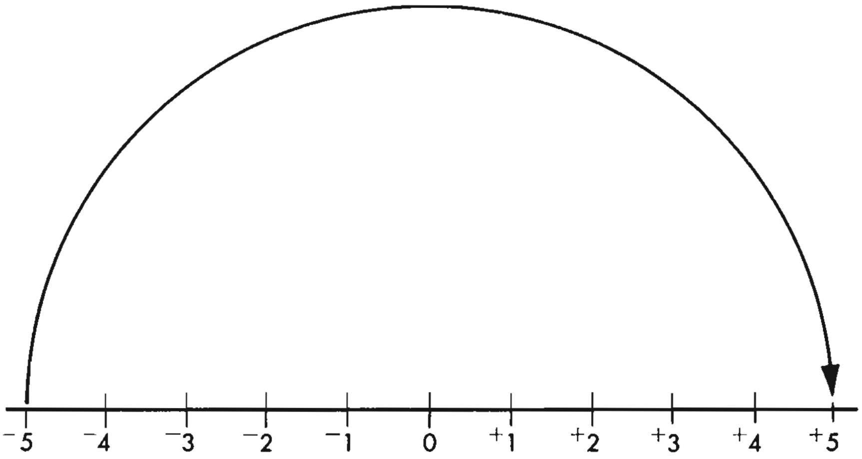
Answers and Comments

(1) **Jerry is right.**

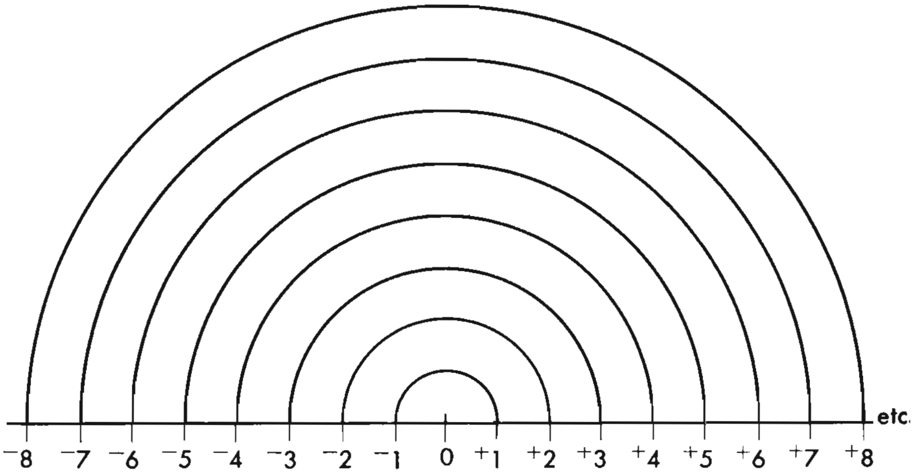
(2)



(3)



A complete rainbow picture for opposites looks like this:



(4) What is $^{+}5 + {}^{-}5 = \_\_\_?$	**(4) 0**
(5) What is the truth set for $^{+}7 + \square = 0?$	**(5) $\{^{-}7\}$**
(6) What is the truth set for $^{-}12 + \square = 0?$	**(6) $\{^{+}12\}$; we could also write this $\{^{\circ}(^{-}12)\}$.**
(7) Ray says this is an identity: $\square + {}^{\circ}\square = 0.$ Do you agree?	**(7) Yes**
(8) John says that any *subtraction* problem can be turned into an *addition* problem by using a certain identity: $\square - \triangle = \_\_\_\_\_\_\_\_.$ Can you guess what identity John uses?	**(8) $\square - \triangle = \square + {}^{\circ}\triangle$**
(9) Can you write down the list of axioms, including two new ones?	**(9) By now, the list of axioms your children will write is probably about like this:** $\square = \square$ **Trivial** $\square + \triangle = \triangle + \square$ **CLA** $\square \times \triangle = \triangle \times \square$ **CLM** $\square \times (\triangle + \bigtriangledown) = (\square \times \triangle) + (\square \times \bigtriangledown)$ **DL** $\square + (\triangle + \bigtriangledown) = (\square + \triangle) + \bigtriangledown$ **ALA** $\square \times (\triangle \times \bigtriangledown) = (\square \times \triangle) \times \bigtriangledown$ **ALM** $\frac{\square}{\square} = 1$ $\square \times 1 = \square$ **L1** $\square + 0 = \square$ **ALZ** $\square \times 0 = 0$ **MLA** $\square + {}^{\circ}\square = 0$ **L Opp** $\square - \triangle = \square + {}^{\circ}\triangle$ **CS** $1 + 1 = 2$ $2 + 1 = 3$ **CN** $3 + 1 = 4$ $\vdots$
(10) Do you know the *names* of these new axioms?	**(10) See answer to problem 9.**

Can you make a derivation for each theorem?

(11) $3 - 2 = 1$

(11) **Theorem:** $3 - 2 = 1$

Proof:

$$\begin{aligned}
3 - 2 &= 3 - 2 && \text{CS}\\
3 - 2 &= 3 + {}^{\circ}(2) && \text{CN}\\
3 - 2 &= (2 + 1) + {}^{\circ}(2) && \text{CLA}\\
3 - 2 &= (1 + 2) + {}^{\circ}2 && \text{ALA}\\
3 - 2 &= 1 + (2 + {}^{\circ}2) && \text{L Opp}\\
3 - 2 &= 1 + 0 && \text{ALZ}\\
3 - 2 &= 1
\end{aligned}$$

Q.E.D.

(12) $(A + B) - B = A$

(12) **Theorem:** $(A + B) - B = A$

Proof:

$$\begin{aligned}
(A + B) - B &= (A + B) - B && \text{CS}\\
(A + B) - B &= (A + B) + {}^{\circ}B && \text{ALA}\\
(A + B) - B &= A + (B + {}^{\circ}B) && \text{L Opp}\\
(A + B) - B &= A + 0 && \text{ALZ}\\
(A + B) - B &= A
\end{aligned}$$

Q.E.D.

(13) $(3 \times \square) + (2 \times \square) = 5 \times \square$

(13) **Theorem:** $(3 \times \square) + (2 \times \square) = 5 \times \square$

Proof:

$$\begin{aligned}
(3 \times \square) + (2 \times \square) &= (3 \times \square) + (2 \times \square) && \text{CLM}\\
(3 \times \square) + (2 \times \square) &= (\square \times 3) + (2 \times \square) && \text{CLM}\\
(3 \times \square) + (2 \times \square) &= (\square \times 3) + (\square \times 2) && \text{DL}\\
(3 \times \square) + (2 \times \square) &= \square \times (3 + 2) && \text{CN}\\
(3 \times \square) + (2 \times \square) &= \square \times [3 + (1 + 1)] && \text{ALA}\\
(3 \times \square) + (2 \times \square) &= \square \times [(3 + 1) + 1] && \text{CN}\\
(3 \times \square) + (2 \times \square) &= \square \times [4 + 1] && \text{CN}\\
(3 \times \square) + (2 \times \square) &= \square \times 5 && \text{CLM}\\
(3 \times \square) + (2 \times \square) &= 5 \times \square
\end{aligned}$$

Q.E.D.

In some later lessons, after the children have become proficient at making this kind of proof, the idea of using any arithmetic fact, or any arithmetic algorithm, is introduced.

If (as in some future lessons) use of a fact of arithmetic is allowed, this same proof would look like this:

Theorem: $(3 \times \Box) + (2 \times \Box) = 5 \times \Box$

Proof:

$$
\begin{aligned}
(3 \times \Box) + (2 \times \Box) &= (3 \times \Box) + (2 \times \Box) && \\
& && \text{CLM} \\
(3 \times \Box) + (2 \times \Box) &= (\Box \times 3) + (2 \times \Box) && \\
& && \text{CLM} \\
(3 \times \Box) + (2 \times \Box) &= (\Box \times 3) + (\Box \times 2) && \\
& && \text{DL} \\
(3 \times \Box) + (2 \times \Box) &= \Box \times (3 + 2) && \\
& && \text{Arithmetic fact} \\
(3 \times \Box) + (2 \times \Box) &= \Box \times 5 && \\
& && \text{CLM} \\
(3 \times \Box) + (2 \times \Box) &= 5 \times \Box &&
\end{aligned}
$$

Q.E.D.

(14) $(A \times \Box) + (B \times \Box) = (A + B) \times \Box$

(14) **Similar to question 13, but somewhat shorter.**

(15) $(3 \times A) + (2 \times A) = 5 \times A$

(15) **Identical to question 13; merely written in a different notation.**

[page 81]

(16) How would you opposite this product?

$${}^{-}2 \times {}^{-}3$$

(16) **The class discussion *may* follow the general lines of questions 17 to 19. Of course, the final *right* answer, which the class will (hopefully) arrive at, is that *you opposite one factor in the product.*** *

(17) Andy says you would opposite a product

$${}^{-}2 \times {}^{-}3$$

by oppositing each factor

$${}^{\circ}({}^{-}2) \times {}^{\circ}({}^{-}3),$$

that is,

$${}^{\circ}(A \times B) = {}^{\circ}A \times {}^{\circ}B.$$

Do you agree?

(17) **No.**

Andy's rule, of course, is *wrong.*
If we opposite *both* factors in ${}^{-}2 \times {}^{-}3 = {}^{+}6$, we get ${}^{+}2 \times {}^{+}3$, which is *still* ${}^{+}6$. In other words, the product (or answer) ${}^{+}6$ *has not been changed.*

To opposite a product correctly, we want to opposite *one* factor; e.g., in ${}^{-}2 \times {}^{-}3$ we can opposite ${}^{-}2$, to get ${}^{+}2 \times {}^{-}3$. This time we succeeded (since ${}^{-}2 \times {}^{-}3 = {}^{+}6$ and ${}^{+}2 \times {}^{-}3 = {}^{-}6$), and ${}^{+}6$ and ${}^{-}6$ are *opposites*, as desired.
Evidently, then, ${}^{\circ}(A \times B) = {}^{\circ}A \times {}^{\circ}B$ is *false;* the correct theorem is ${}^{\circ}(A \times B) = ({}^{\circ}A) \times B$ (i.e., B has *not* been opposited).

(18) Can you state a rule:

${}^{\circ}(A \times B) =$ ______?

(18) **${}^{\circ}(A \times B) = ({}^{\circ}A) \times B$ or ${}^{\circ}(A \times B) = A \times {}^{\circ}B$**

---

* A classroom presentation of questions 16 through 19 is available on a tape recording.

(19) See if you can make a derivation for your theorem in question 18.

(19) **Theorem:** ${}^{\circ}(A \times B) = ({}^{\circ}A) \times B$

Proof:

$${}^{\circ}(A \times B) = {}^{\circ}(A \times B)$$

ALZ

$${}^{\circ}(A \times B) = {}^{\circ}(A \times B) + 0$$

MLZ

$${}^{\circ}(A \times B) = {}^{\circ}(A \times B) + (0 \times B)$$

L Opp

$${}^{\circ}(A \times B) = {}^{\circ}(A \times B) + [(A + {}^{\circ}A) \times B]$$

DL and* CLM

$${}^{\circ}(A \times B) = {}^{\circ}(A \times B) + [(A \times B) + \{({}^{\circ}A) \times B\}]$$

ALA

$${}^{\circ}(A \times B) = [{}^{\circ}(A \times B) + (A \times B)] + ({}^{\circ}A) \times B$$

CLA

$${}^{\circ}(A \times B) = [(A \times B) + {}^{\circ}(A \times B)] + ({}^{\circ}A) \times B$$

L Opp

$${}^{\circ}(A \times B) = 0 + ({}^{\circ}A) \times B$$

CLA

$${}^{\circ}(A \times B) = ({}^{\circ}A) \times B + 0$$

ALZ

$${}^{\circ}(A \times B) = ({}^{\circ}A) \times B$$

Q.E.D.†

This is a very difficult derivation. It seems amazing that moderately bright fifth graders *can*, under good conditions, conjecture this theorem *and prove it!*

Most teachers, with most classes, will be well advised to omit this derivation. You may still want to let your children *conjecture* the theorem, even if you omit proving it.

---

* Several steps were taken at once at this point. Sooner or later the children are ready to do this.

† On the tape-recorded lesson, the children give a derivation slightly different from the one given here.

chapter 41 / *Pages 81–83 of Student Discussion Guide*

SOME MORE MACHINES

Chapter 41

SOME MORE MACHINES

Can you make a "machine" or *formula* that will give the truth set for each equation? [page 81]

(1) $\square + a + b + c = r + s + u$

(2) $\square + a - (b + c) = r + s + u$

(3) $\square + a - b + c = r + s + u$

(4) $\square + a + b + c = r + s - u$

(5) $\square + a + b + c = r - s + u$

(6) $\square + a + b + c = r - (s + u)$

(7) $(\square \times a) + (\square \times b) = c$

(8) $(\square \times a) + (\square \times 9) = c$

(9) $(\square \times a) + \square = c$

(10) $\square + \square + \square = c$

ANSWERS AND COMMENTS

(1) $\square = (r + s + u) - (a + b + c)$

or

$\square = r + s + u - a - b - c$

(2) $\square = [r + s + u] - [a - (b + c)]$

or

$\square = (r + s + u) - (a - b - c)$

or

$\square = r + s + u - a + b + c$

etc.

(3)
(4)
(5)
(6) **Similar to problems 1 and 2.**

(7) $\square = \dfrac{c}{a + b}$ **An old friend! If necessary, use the distributive law before trying to find the machine.**

(8) $\square = \dfrac{c}{a + 9}$

(9) $\square = \dfrac{c}{a + 1}$ **Another old friend! If necessary, use the law for one and the distributive law before trying to find the machine.**

(10) **The open sentence $\square + \square + \square = 3 \times \square$ is an identity. Therefore, we have**

$$3 \times \square = c$$

and

$$\square = \frac{c}{3}.$$

[page 82]

(11) Do you know what we mean by 3^2?	(11) $3 \times 3 = 9$
(12) Can you find each value? (a) 4^2 (b) 5^2 (c) 11^2 (d) 2^2 (e) 2^3 (f) $({}^-5)^2$ (g) $({}^+6)^2$ (h) $({}^-10)^2$	(12) (a) $4 \times 4 = 16$ (b) $5 \times 5 = 25$ (c) $11 \times 11 = 121$ (d) $2 \times 2 = 4$ (e) $2 \times 2 \times 2 = 8$ (Note: three factors) (f) $({}^-5) \times ({}^-5) = {}^+25$ (g) 36 (h) ${}^+100$
(13) Can you find each truth set? (a) $\square \times \square = 16$ (b) $\square^2 = 25$ (c) $\square^2 = 169$ (d) $\square^2 = 196$ (e) $\square^2 = 4$ (f) $\square^2 = {}^-100$	(13) (a) $\{{}^+4, {}^-4\}$ (b) $\{{}^+5, {}^-5\}$ (c) $\{{}^+13, {}^-13\}$ (d) $\{{}^+14, {}^-14\}$ (e) $\{{}^+2, {}^-2\}$ (f) {▨} (i.e., there is *no* real number whose square is ${}^-100$).
(14) Do you know what we mean by $\sqrt{}$?	(14) This is left for you to explain to your students. (The question, of course, is rhetorical.)
(15) Can you find each positive square root? (a) $\sqrt{16} = ?$ (b) $\sqrt{49} = ?$ (c) $\sqrt{9} = ?$ (d) $\sqrt{900} = ?$ (e) $\sqrt{625} = ?$	(15) (a) ${}^+4$ (b) ${}^+7$ (c) 3 (d) 30 (e) 25
(16) Suppose that S is a perfect square. Can you make up a machine to solve $\square^2 = S$?	(16) The truth set of $\square^2 = s$, where s is a perfect square, is of course $\{\sqrt{s}, {}^{\circ}\sqrt{s}\}$. Two machines are needed (one for each "answer"): $\square = \sqrt{s}$ and $\square = {}^{\circ}\sqrt{s}$.
(17) Al says there are two solutions for $\square^2 = S$. He has made up two separate machines. One machine gives him one solution, and the other machine gives him the other. Can you guess what machine Al uses?	(17) See the answer to question 16.

[page 83]

(18) Nancy has been working on derivations. She has made up a derivation for the theorem

$${}^{\circ}(A+B) = {}^{\circ}A + {}^{\circ}B.$$

Can you make up a derivation for this theorem?

(18) **This is another hard, but important, theorem.**

Theorem: $\mathbf{{}^{\circ}(A+B) = {}^{\circ}A + {}^{\circ}B}$

Proof:

$${}^{\circ}(A+B) = {}^{\circ}(A+B)$$

ALZ

$${}^{\circ}(A+B) = {}^{\circ}(A+B) + 0$$

L Opp

$${}^{\circ}(A+B) = {}^{\circ}(A+B) + (A + {}^{\circ}A)$$

ALZ*

$${}^{\circ}(A+B) = {}^{\circ}(A+B) + (A + {}^{\circ}A) + 0$$

L Opp

$${}^{\circ}(A+B) = {}^{\circ}(A+B) + (A + {}^{\circ}A) + (B + {}^{\circ}B)$$

†

$${}^{\circ}(A+B) = {}^{\circ}(A+B) + (A + B) + ({}^{\circ}A + {}^{\circ}B)$$

ALA

$${}^{\circ}(A+B) = [{}^{\circ}(A+B) + (A + B)] + ({}^{\circ}A + {}^{\circ}B)$$

CLA

$${}^{\circ}(A+B) = [(A + B) + {}^{\circ}(A+B)] + ({}^{\circ}A + {}^{\circ}B)$$

L Opp

$${}^{\circ}(A+B) = 0 + ({}^{\circ}A + {}^{\circ}B)$$

CLA

$${}^{\circ}(A+B) = ({}^{\circ}A + {}^{\circ}B) + 0$$

ALZ

$${}^{\circ}(A+B) = {}^{\circ}A + {}^{\circ}B$$

Q.E.D.

Work similar to problem 18 is only for good students.

The purpose of this *Student Discussion Guide* is to give the children *experience* with algebraic and mathematical thinking. It does *not* hold to predetermined levels of achievement for *all* fifth graders. Problems such as 18 should be used only with children for whom you feel *this* kind of experience is appropriate. As you can hear on Madison Project tape recordings, this type of mathematical experience *is* appropriate for many bright fifth graders, but problems such as 18 are *beyond* the core level of Madison Project material. With most classes, it is not advisable to strive for this level of sophistication. There is plenty of important work at a lower level.

---

* This step, for simplicity, uses the generalized associative property for addition. You can fill in each step meticulously if this casual attitude is offensive. (Of course, one must know *when* to be casual, and *when* to be meticulous!)

† Some more use of the general associative and commutative properties for addition. Because associative and commutative laws for addition are so "reliable," we can *afford* to be more casual with them than we can with an "unreliable" identity. The unreliable or tricky identities, such as the distributive law, law of opposites, changing signs, etc., need meticulous attention to detail. Associative and commutative laws for addition usually do not.

chapter 42 / *Pages 83–84 of Student Discussion Guide*

AREA

Chapter 41 carried the students to the highest level of sophistication in the subject of identities and derivations. It can safely be called one of the towering pinnacles of Madison Project material, and cannot be attained by everyone.

Now, in Chapter 42, a new topic is begun on an extremely simple level. Chapters 42 and 43 can easily be taught to most moderately bright third, fourth, or fifth graders.

The point of this chapter is to get the student to realize that *size* can be measured in many different ways.

What is the largest building in town? Well, this may mean the *tallest*, or the one with the most imposing facade, or the one with the most square feet of floor space, or the one with the longest frontage, or the one with the most rooms, or the one with the largest volume, or the one with the most floors, or the one with the tallest chimney or flagpole or spire . . . . These may all be *different* buildings.

How *large* is a geometric figure? *There are many different criteria of size for geometric figures:* length, topological diameter, perimeter, area and volume, girth.

This chapter is included because children do not realize that, in developing mathematical "measures," we start with some *unverbalized* intuitive idea of what *kind* of "bigness" is important in the case at hand, and then gradually (and more-or-less arbitrarily) develop some abstract and precise criterion as a refinement to (or approximation of) our general intuitive idea of "bigness" or "littleness."

If children do not learn measurement in these terms, this is an ironic tragedy, for the advanced mathematician approaches the subject this way—and *surely the child, if allowed to proceed on his own, does also.*

Chapter 42
AREA

ANSWERS AND COMMENTS

[page 83]

(1) Who is the biggest student in the class?	(1) **It all depends on how you go about defining *biggest*. See question 2.**
(2) John says it all depends. Do you mean the tallest, or the heaviest, or the oldest, or the strongest? What do you think?	(2) **John is right.**
(3) Which is bigger, a pocket jackknife, or a piece of paper?	(3) **It all depends on how you choose to define *bigger*.**
(4) Which is bigger, an automobile or a telephone pole?	(4) **Similar to question 3.**
(5) Do you know what we mean by **length?**	(5) **The idea of length is to take some *unit*, such as an *inch* segment or a *centimeter* segment etc., and see "how many times the unit segment will fit into the segment to**

be measured." (You or your children may find even better ways of saying what is meant by *length*.)

(6) Do you know what we mean by **area**?

(6) Similar to question 5, but the unit is now a *unit square*. (Or, a triangular area could be defined by using a *unit triangle*, for example, an equilateral triangle one unit along each edge. The resulting area measure of plane figures would be quite different from the familiar measure based upon a square unit. This triangular measure is discussed in some recent Russian books on geometry.)

(7) Do you know what we mean by **volume**?

(7) Similar to question 6, but the unit is now a *unit cube*.

(8) Can you think of something that is very **tall**, but has a small **volume**?

(8) A flagpole might be one example.

(9) Can you think of something that has a big **area**, but has a small **volume**?

(9) A page of a newspaper might be one example.

(10) Can you think of something that has a small **area**, but a larger **volume**?

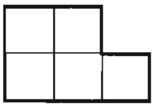
(10) A baseball might be one example.

There is an excellent opportunity at this point (problems 11, 12, and 13, and in Chapter 43) to introduce, if you wish, the idea of **congruent** *figures. You may want to distinguish congruence in two senses: allowing only motion within the plane or allowing also flipping figures over by lifting them momentarily out of the plane.*

[page 84]

(11) Now, using a piece of graph paper, see if you can draw a figure that has an area of 5 squares.

(11) Here are a few possibilities:



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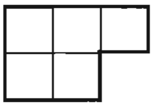

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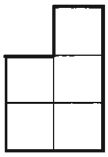

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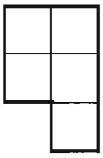

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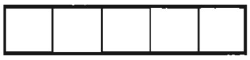

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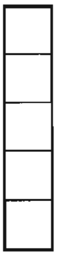

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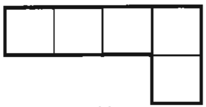

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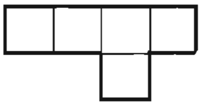

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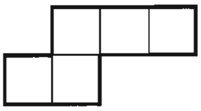

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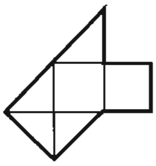

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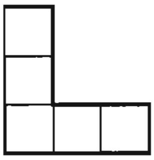

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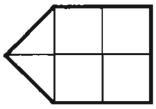

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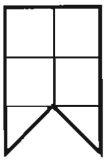

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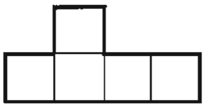

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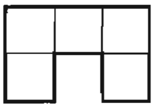

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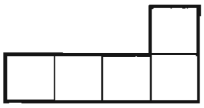


Q

If you are interested in introducing the idea of congruent geometric figures at this stage, you might notice the relations among these figures.

If we allow motion only in the plane, *without lifting the figure out of the plane and flipping it over,* then: A and C are congruent; B and D are congruent; E and F are congruent; and I and O are congruent.

If we allow motion in the plane, and also allow the figures to be lifted out of the plane and flipped over, then: A, B, C, and D are congruent; E and F are congruent; H and Q are congruent; and I and O are congruent.

(12) Al drew this picture.

Does it have an area of 5 squares?

(12) **Yes**

(13) Did anyone in your class draw a picture **different** from the one that Al drew? Did his picture have an area of 5 squares?

(13) **See the answer to question 11.**

chapter 43 / *Pages 84–87 of Student Discussion Guide*

HOW MANY SQUARES?

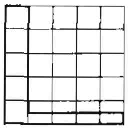
This is a pleasant and easy lesson, and very suitable for use with younger children. Its objective might be described as building readiness for further work in *area*.

Chapter 43
HOW MANY SQUARES?

[page 84]

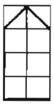
(1) Can you draw a figure that has an area of 7 squares?

(2) Debbie drew this picture. Does it have an area of 7 squares?



(3) Marc drew this picture. Does it have an area of 7 squares?

(4) Ellen drew this picture. What area does it have?

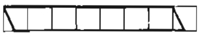


[page 85]

(5) Jerry drew this picture. What area does it have?

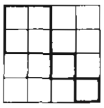
(6) Tony drew this picture. What area does it have?

(7) Alec drew this picture. What area does it have?



What is the area of each figure?

(8)



ANSWERS AND COMMENTS

(1) There are many possibilities. (See the answer to question 12 in Chapter 42.)

(2) Yes. Five whole squares, and four more half squares, for a total of seven.

(3) Yes

(4) Seven squares

(5) Seven squares

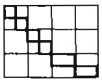
(6) Eight squares

(7) Seven squares (note how the "extra" piece on the right just fills in for the "missing" piece on the left).

(8) Six squares

(9)


(9) Two squares

(10)


(10) Two squares

(11)


(11) Four squares

(12)


(12) Four squares

(13)

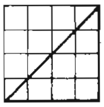

(13) Seven squares

(14)


(14) Three squares

(15)


(15) Five squares

(16)


(16) Eight squares

(17)


(17) Two squares

(18)


(18) One-half square

(19)


(19) One-half square

[page 86]

(20) Jerry says this figure has an area of 5 squares. Do you agree?

(20) Jerry is right. (Notice that the triangular-shaped piece at the left end is one half of two squares; hence, it has an area equivalent to one square.)

(21) Can you draw a figure with an area of 4 squares?

(21) There are many, many possibilities.

(22) Can you draw a figure with an area of $2\frac{1}{2}$ squares?

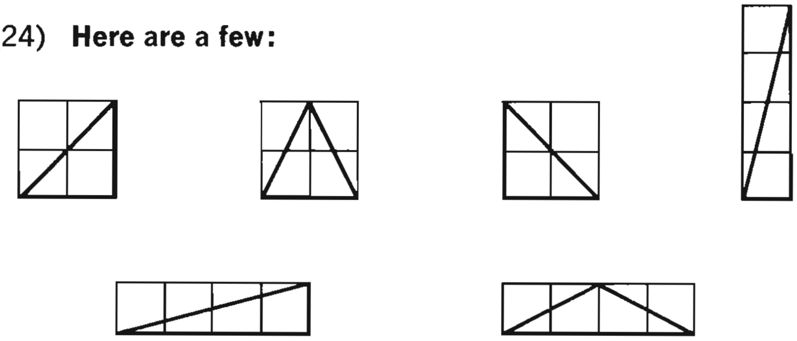
(22) There are many, many possibilities.

(23) Do you know what a **triangle** is?

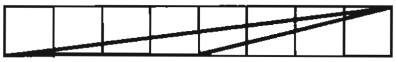
(23) The children can perhaps best answer this by drawing several triangles.

(24) Can you draw a triangle that has an area of 2 squares?

(24) Here are a few:



Here is a very tricky one; it is advisable to use it only with bright, somewhat older children (fifth, sixth, or seventh graders, or older):

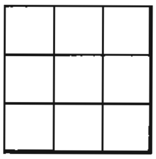


(25) Do you know what a **rectangle** is? Do you know why we use the name **rectangle?**

(25) The children can probably best answer this by drawing a few rectangles.

(26) What is a **right angle?** Can you draw a right angle?

(26)



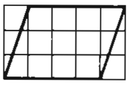
(27) Can you draw a rectangle that has an area of 8 squares?

(27) Even here there are many possibilities.

(28) Do you know what a **parallelogram** is?

(28) The children can best answer this by drawing a few. You are likely to have to *show* them, since they may not know.

(29) Jerry says this is a parallelogram. Do you agree?



(29) Jerry is right.

(30) Can you draw a parallelogram that has an area of 5 squares?

(30) Here is one:

(31) Do you know what a **trapezoid** is?

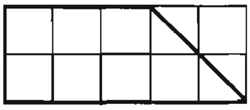
(31) You will probably need to tell the children; the question is really rhetorical.

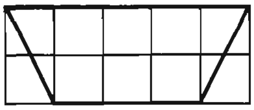
(32) Hal says that a trapezoid has two parallel sides. Is this a good description of a trapezoid?

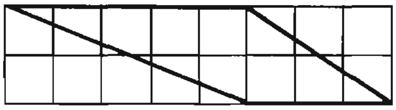
(32) Almost, but not quite. If you add the fact that a trapezoid has four sides, then Hal will be right. (See question 34.)

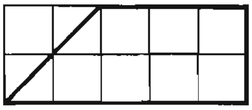
(33) Can you draw a trapezoid that has an area of 8 squares?

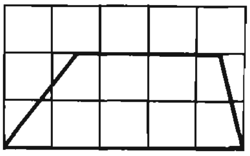
(33) See questions 34 through 46. A "correct" answer here would be something like one of these:

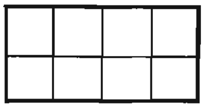


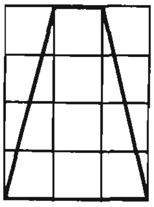


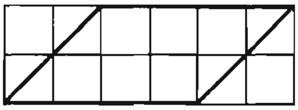




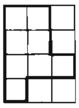








(34) Aaron drew this picture.



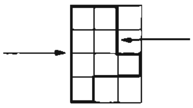
Is this a trapezoid with an area of 8 squares?

(34) No. Aaron's picture is *not* a trapezoid, because it is not a four-sided figure.

(35) Jerry says that Aaron's figure is not a trapezoid. What do you think?

(35) Jerry is right. See answer to question 34.

(36) Aaron says his figure has these two parallel sides, so it *must* be a trapezoid.



(36) See answers to questions 32, 34, and 35.

(37) Hal says that what he meant to say was that a trapezoid is a **four-sided** figure with two parallel sides. What do you think?

(37) *Now* Hal is right!

It is probably best to omit questions 38 through 43 with slower students. This idea of **subsets** *might only confuse them.*

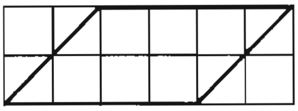
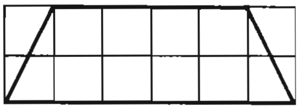
[page 87]

(38) Is a rectangle also a trapezoid?

(38) Yes, since it *is* a four-sided figure with (at least) two parallel sides (actually, of course, with two *pairs* of parallel sides, and four right angles thrown in as a bonus!).

(39) Is a parallelogram also a trapezoid?

(39) Yes. Just as a cow is an animal, but an animal is *not* necessarily a cow.

(40) Is every trapezoid also a rectangle?	**(40) No. An animal need *not* be a cow, even though every cow is an animal.**
(41) Is every trapezoid also a parallelogram?	**(41) No, not necessarily.**
(42) Is every square also a rectangle?	**(42) Yes**
(43) Is every square also a parallelogram?	**(43) Yes**
See if you can draw these figures:	
(44) A trapezoid with an area of 6 squares.	**(44) There are many possibilities.**
(45) A trapezoid that is also a parallelogram.	(45) 
(46) A trapezoid that is **not** a parallelogram.	(46) 
(47) A triangle that has an area of $3\frac{1}{2}$ squares.	**(47) There are many possibilities.**
(48) A figure that has an area of $\frac{1}{4}$ square.	**(48) There are many possibilities.**
(49) A rectangle that has an area of 60 squares.	**(49) There are many possibilities.**
(50) A triangle that has an area of 30 squares.	**(50) There are many possibilities.**

chapter 44 / *Pages 87–89 of Student Discussion Guide*

WORD PROBLEMS*

This chapter is probably largely self-explanatory as to its objectives, and so on, but one comment is worth making.

In some of the earliest attempts at teaching word problems to fifth and eighth graders, the Madison Project used some "traditional" materials—which meant, primarily, using problems that lead to simple linear equations. This did not work satisfactorily. The children could invariably find the answer by arithmetic alone, without recourse to algebra. Consequently, the more powerful methods of algebra remained a closed book to them, although it was precisely these methods that we wanted the children to learn.

We felt (rather strongly) that it would probably be injudicious to *tell* the children that they *must* solve the problems by algebraic methods. The task, rather, was to structure the situation so that the children would *want* to use algebra.

The following problems resolve the difficulty very nicely. These problems become *very* much simpler when approached by algebraic techniques. Since using this new assortment of word problems, the results have been very satisfying.

Chapter 44
WORD PROBLEMS

[page 87]

The use of placeholders often helps us to solve problems.

(1) Mr. Wilson gets on a streetcar at the corner near his house. No one else gets on at this stop. At the next stop two people get on. At the third stop, three people get on and one man gets off. At the fourth stop, one man gets on and two get off. At the next stop five people get on. At the sixth stop, one half of the passengers on the car get off. At the next stop, Mr. Wilson again notices that one half of the passengers get off. At the eighth stop three passengers get off, and Mr. Wilson is the only passenger left on the car.

How many passengers were on the car when Mr. Wilson got on?

ANSWERS AND COMMENTS

(1) Seven

The essential algebraic technique here is, of course, the use of *placeholder* or *variable* notation. That is to say, letting P stand for the number of passengers on the streetcar before Mr. Wilson got on, we can make the following table:

	Number of Passengers
Before Mr. Wilson got on:	P
After Mr. Wilson got on:	$P + 1$
After the second stop:	$P + 3$
After the third stop:	$P + 5$
After the fourth stop:	$P + 4$
After the fifth stop:	$P + 9$
After the sixth stop:	$\frac{1}{2} \times (P + 9)$
After the seventh stop:	$\frac{1}{4} \times (P + 9)$
After the eighth stop:	$[\frac{1}{4} \times (P + 9)] - 3$

Now, at this point Mr. Wilson is the only passenger left on the car, so . . . $[\frac{1}{4} \times (P + 9)] - 3 = 1$

---

* A tape recording of this lesson is available.

At this stage, a second algebraic technique comes into play, namely, the use of *transform operations.* We can proceed as follows:

$$\begin{aligned}[\tfrac{1}{4} \times (P+9)] - 3 &= 1\\ \tfrac{1}{4} \times (P+9) &= 4\\ P+9 &= 16\\ P &= 7\end{aligned}$$

Thus we see that there must have been seven passengers already on the streetcar at the time Mr. Wilson got on.

(2) Larry used a placeholder to solve problem 1.

	Number of Passengers
Before Mr. Wilson got on:	P
After Mr. Wilson got on:	$P + 1$
After second stop:	$P + 3$
After third stop:	$P + 5$
After fourth stop:	$P + 4$ **[page 88]**
After fifth stop:	$P + 9$
After sixth stop:	$P + 4\frac{1}{2}$

(2) There is of course, an error in Larry's work on the line labeled "After the sixth stop." See the answer to problem 1; notice that what is involved here is actually the *distributive law.*

(3) At this point Debbie interrupted. She says there can't be $4\frac{1}{2}$ people. What do you think?

(3) Debbie is right. See the answer to problem 2.

(4) Cindy says that after the fifth stop, there were

$P + 9$ people,

so after the sixth stop there were

$\frac{1}{2} \times (P + 9)$ people.

What do you think?

(4) Cindy is right.

(5) See if you can finish Larry's solution.

(5) See the answer to problem 1.

(6) On Tuesday Jerry gets twice as much allowance as he gets on Monday. On Wednesday he gets twice as much as he does on Tuesday. On Thursday he gets 50 cents. On Friday he gets twice as much as he got on Wednesday. On Saturdays and Sundays he gets nothing. For the entire week he gets $2.15. How much does he get on Monday?

(6) 11 cents

Again (as in problem 1), the decisive technique is the use of a "placeholder" (or "variable," or "unknown"), such as the box.

Suppose we use the box to represent (or "hold a place for") the amount of money Jerry gets on Monday (say, the number of cents he gets on Monday).

Then, on Tuesday, he gets $(2 \times \square)$ cents.
On Wednesday, he gets $2 \times (2 \times \square) = 4 \times \square$ cents.
On Thursday, he gets 50 cents.
On Friday, he gets $8 \times \square$ cents.
Adding up, we see that Jerry gets

$$\square + (2 \times \square) + (4 \times \square) + 50 + (8 \times \square) \text{ cents}$$

each week. We know, however, that this amount is 215 cents. Hence we have the equation:

$$\square + (2 \times \square) + (4 \times \square) + 50 + (8 \times \square) = 215.$$

We can simplify this greatly by our fifth transform operation, namely, the use of identities. Without going through all the details, we can use identities to obtain the equivalent equation

$$\left(15 \times \square\right) + 50 = 215.$$

We can now subtract 50 from each side, to obtain the equivalent equation $15 \times \square = 165$.

For this last equation, however, the truth set is $\{11\}$; hence, Jerry must get 11 cents on Monday.

(7) A few days later, Mr. Wilson again counted passengers on the streetcar as he rode to work in the morning.

Mr. Wilson and one other man got on at the same stop. (The other man almost missed the streetcar. He came running up just as it was about to pull away.)

At the second stop, three people got on.

At the third stop, two people got on and one got off.

At the fourth stop, three people got off.

At the fifth stop, six people got on and two got off.

At the sixth stop, one half of the passengers got off.

At the seventh stop, four passengers got off, and Mr. Wilson was the only passenger left on the streetcar.

How many people were on the streetcar when Mr. Wilson got on?

(7) **This is generally similar to Mr. Wilson's previous adventures in problem 1.**

(8) Alec: "Think of a number . . ."
Ellen: "All right."
Alec: ". . . multiply it by itself . . ."
Ellen: "All right."
Alec: ". . . subtract 4 times the original number . . ."
Ellen: "All right."
Alec: ". . . add 10 . . ."
Ellen: "All right."
Alec: ". . . and tell me the answer."
Ellen: "15." [page 89]
Alec: "The number you started with was 5."
Ellen: "Oh no it wasn't!"

What number did Ellen use?

(8) **$^{-}1$**

The decisive technique is the use of the box as "the number":
Alec:

"Think of a number . . . $\square$
"multiply it by itself . . . $\square \times \square$
"subtract 4 times the original number . . . $\left(\square \times \square\right) - \left(4 \times \square\right)$
"add ten . . . $\left(\square \times \square\right) - \left(4 \times \square\right) + 10$
"tell me the answer."
Ellen:
"Fifteen." $\left.\right\}$ $\left(\square \times \square\right) - \left(4 \times \square\right) + 10 = 15$

In normal form, this equation would be

$$\left(\square \times \square\right) - \left(^{+}4 \times \square\right) + {}^{-}5 = 0.$$

The truth set is $\{^{+}5, {}^{-}1\}$.
Since Ellen did not use 5, she must have used $^{-}1$.

(9) The height of a certain triangle is 21 feet greater than the length of its base. If the area is 50 square feet, what are the dimensions of the triangle?

(9) **Base: 4 feet**
Height: 25 feet

Let b denote the length of the base, in feet. Then the height must be $b + 21$. The area would then be $\frac{1}{2} \times (b) \times (b + 21)$, and we get the equation

$$\tfrac{1}{2} \times b \times (b + 21) = 50,$$

which is equivalent to the following:

$$\begin{gathered} b \times (b + 21) = 100 \\ (b \times b) + (21 \times b) - 100 = 0 \\ (b \times b) - (^{-}21 \times b) + {}^{-}100 = 0. \end{gathered}$$

For this last equation, the truth set is $\{^{-}25, {}^{+}4\}$. The length of the base of a triangle cannot be negative; hence, it must be $^{+}4$.

(10) John has more marbles than Henry does. Henry has more than twice as many marbles as Albert has. One fourth of Albert's marbles were given to him by his father, and the other three fourths are some large green ones that he bought himself. John has less than eleven marbles.

How many marbles does each boy have?

(10) **John has 10; Henry has 9; Albert has 4.**

We can get the following system of equations and inequalities:

$H < J$

$(2 \times A) < H$

$4 \mid A$ (This notation means that "4 divides A", i.e., A is an integer multiple of 4, such as 0, 4, 8, 12, 16, 20, . . .)

$J < 11$

Combining, we get $8 \le (2 \times A) < H < J < 11$. In order to get a true statement, the only numbers that we can insert into this inequality at these points

$$8 \le \underset{\uparrow}{(2 \times A)} < \underset{\uparrow}{H} < \underset{\uparrow}{J} < 11,$$

are, evidently,

$$8 \le \underset{\underset{8}{\uparrow}}{(2 \times A)} < \underset{\underset{9}{\uparrow}}{H} < \underset{\underset{10}{\uparrow}}{J} < 11.$$

Consequently, $A = 4$, $H = 9$, $J = 10$.

(11) Last year the Lincoln Junior High School Ski Club went skiing on weekends. Every weekend except the fourth, two thirds of those who went were injured. Each member went every weekend until he was injured; after that he did not go again.

At the beginning of the season, the club bought 100 tickets on the ski tow. Each ticket was good for one member for one weekend. After the fourth weekend, the club had 20 tickets which they had not yet used. By unanimous vote of the members, the club sold these 20 tickets to the Levy Ski Club and changed the name to the Lincoln Junior High School Bowling Club.

How many members of the Lincoln Junior High School Ski Club went skiing on the first weekend?

(11) **54**

Number who went skiing on the first weekend: $\Box$

Number who went skiing on the second weekend: $\frac{1}{3} \times \Box$

Number who went skiing on the third weekend: $\frac{1}{3} \times \left[\frac{1}{3} \times \Box\right]$

i.e., $\frac{1}{9} \times \Box$

Number who sent skiing on the fourth weekend: $\frac{1}{3} \times \left[\frac{1}{9} \times \Box\right]$

i.e., $\frac{1}{27} \times \Box$

Number of tickets used on four weekends:

$$\square + \left(\tfrac{1}{3} \times \square\right) + \left(\tfrac{1}{9} \times \square\right) + \left(\tfrac{1}{27} \times \square\right).$$

We get the equation

$$\square + \left(\tfrac{1}{3} \times \square\right) + \left(\tfrac{1}{9} \times \square\right) + \left(\tfrac{1}{27} \times \square\right) = 80,$$

i.e.,

$$\square \times [1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27}] = 80$$

$$\square \times \left[\frac{27 + 9 + 3 + 1}{27}\right] = 80$$

$$\square \times \tfrac{40}{27} = 80$$

$$\square \times \tfrac{1}{27} = 2$$

$$\square = 2 \times 27$$

and the truth set is $\{54\}$.

chapter 45 / *Pages 90–92 of Student Discussion Guide*
SIMULTANEOUS EQUATIONS

Chapter 45
SIMULTANEOUS EQUATIONS

Can you make a graph for the truth set of each equation?

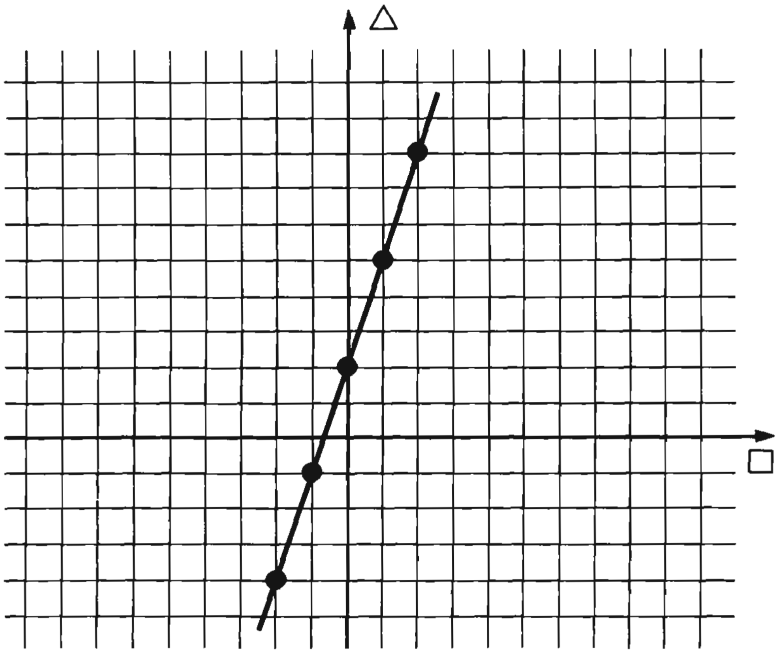
[page 90]

(1) $\triangle = (^{+}3 \times \square) + {}^{+}2$

(2) $\triangle = (^{+}5 \times \square) + {}^{-}3$

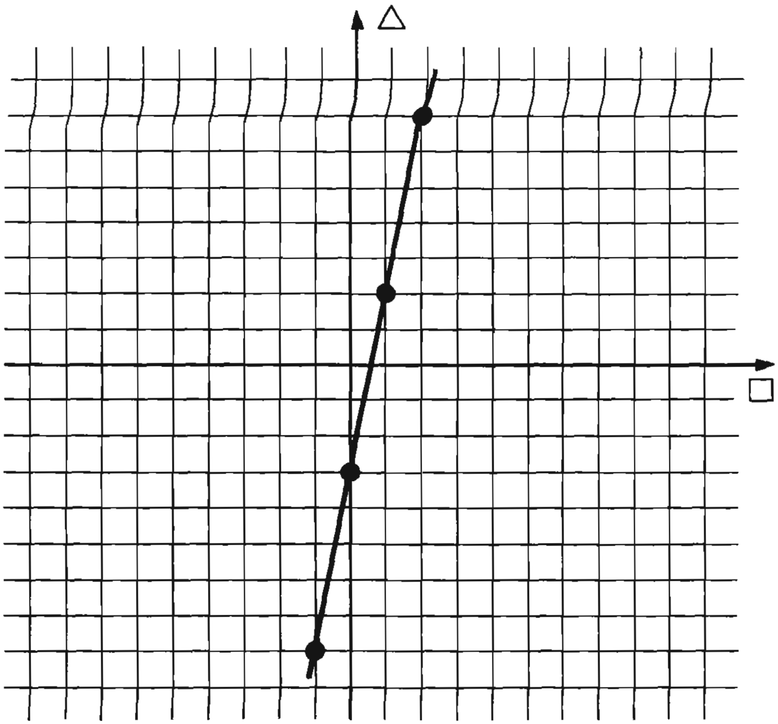
ANSWERS AND COMMENTS

(1)



Graph to indicate truth set of $\triangle = (^{+}3 \times \square) + {}^{+}2$

(2)



Graph to indicate truth set of $\triangle = (^{+}5 \times \square) + {}^{-}3$

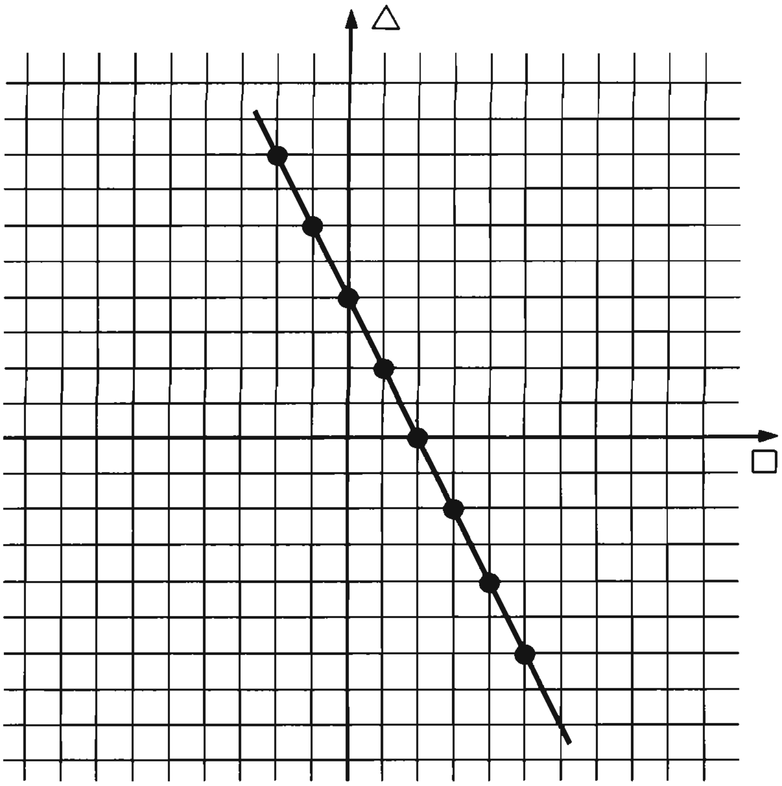
(3) $\triangle = ({}^{-}2 \times \square) + {}^{+}4$

(4) Lex says that he knows that $(\boxed{{}^{+}2}, \overset{\triangle}{{}^{+}8})$ is a point on the line for question 1. Therefore, he says he knows something **else** about $(\boxed{{}^{+}2}, \overset{\triangle}{{}^{+}8})$. What else does Lex know?

(5) Mary says that if you substitute $(\boxed{{}^{+}1}, \overset{\triangle}{{}^{+}2})$ into the equation of question 2, you will get a **true** statement. Does the point $(\boxed{{}^{+}1}, \overset{\triangle}{{}^{+}2})$ lie on the graph for question 2?

Can you find **one** pair of numbers to fill $\square$, $\triangle$ so that **both** statements will be true?

(3)



Graph to indicate truth set of $\triangle = ({}^{-}2 \times \square) + {}^{+}4$

The purpose of questions 4 and 5 is to emphasize that a point lies on the graph of the truth set if and only if its co-ordinates, when substituted into the open sentence, produce a true statement.

To say this another way, the *geometric condition* that:

the point (a, b) lies on the
graph of an equation $y = f(x)$

is precisely equivalent to the *algebraic condition* that:

$b = f(a)$ is a true statement.

This is essentially the idea that is expressed in traditional language by the word *locus.*

Since the geometric and algebraic conditions are fully equivalent, we may work with whichever one we prefer. In simultaneous equations, it will frequently turn out to be much easier to use the *geometric* condition. (You may prefer to approach this with your children via some other method.)

(4) **He knows that if ${}^{+}2$ is substituted in the box and ${}^{+}8$ is substituted in the triangle the result will be a *true* statement.**

(5) **Yes**

(6) $\triangle = ({}^{+}3 \times \square) + {}^{-}2$

$\triangle = ({}^{-}2 \times \square) + {}^{+}8$

(6) $2 \rightarrow \square$

$4 \rightarrow \triangle$

The children may solve this problem by "guessing." As these problems are made more complicated (for example, by introducing fractional roots), the children are induced to invent more systematic methods of solution.

One fifth grader (Lex) suggested listing the truth set for each equation to see which pair of numbers appears on both lists. This is a wonderful idea, but it is made more useful by adding Debbie's suggestion that each truth set be represented by a graph.

(7) $\triangle = ({}^{+}2 \times \square) + {}^{+}1$

$\triangle = ({}^{-}2 \times \square) + {}^{+}5$

(7) $1 \rightarrow \square$

$3 \rightarrow \triangle$

(8) $\triangle = ({}^{+}3 \times \square) + {}^{+}1$

$\triangle = ({}^{-}1 \times \square) + {}^{+}9$

(8) $2 \rightarrow \square$

$7 \rightarrow \triangle$

(9) $\triangle = ({}^{+}4 \times \square) + {}^{+}5$

$\triangle = ({}^{-}2 \times \square) + {}^{+}5$

(9) $0 \rightarrow \square$

$5 \rightarrow \triangle$

(10) Have you found a general method for solving these equations? Can you describe it?

(10) **What is wanted here is the Lex-Debbie method referred to in answer to question 6: graph each equation and look for points of intersection.**

There is no way of predicting what methods, if any, your children may devise.

Can you solve each pair?

(11) $\triangle = ({}^{+}2 \times \square) + {}^{+}1$

$\triangle = ({}^{+}4 \times \square) + {}^{+}4$

(11) $\frac{{}^{-}3}{2} \rightarrow \square$

${}^{-}2 \rightarrow \triangle$

(12) $\triangle = ({}^{+}2 \times \square) + {}^{+}7$

$\triangle = ({}^{+}4 \times \square) + {}^{+}12$

(12) $\frac{{}^{-}5}{2} \rightarrow \square$

${}^{+}2 \rightarrow \triangle$

Now a sequence of questions is begun in the hope that the children will discover the method of *eliminating one unknown, and getting one equation in one unknown.*

(13) $\triangle = \square$

$\triangle = ({}^{+}3 \times \square) + {}^{-}4$

(13) $2 \rightarrow \square$

$2 \rightarrow \triangle$

[page 91]

(14) $\triangle = \square$

$\triangle = ({}^{+}4 \times \square) + {}^{-}3$

(14) ${}^{-}1 \rightarrow \square$

${}^{-}1 \rightarrow \triangle$

(15) $\triangle = \square + 1$

$\triangle = ({}^{+}3 \times \square) + {}^{-}3$

(15) $2 \rightarrow \square$

$3 \rightarrow \triangle$

(16) $\triangle = 2 \times \square$

$\triangle = (4 \times \square) + 7$

(16) $^{-}3\frac{1}{2} \rightarrow \square$

$^{-}7 \rightarrow \triangle$

(17) Sarah has found a second method for solving simultaneous equations.

How many methods have you found?

(17) Sarah's method is to say (using question 14 as an illustration):

Since $\triangle = \square$, we know that $\triangle$ is merely *another name* for $\square$.

In the equation

$$\triangle = ({}^{+}4 \times \square) + {}^{+}3,$$

we replace the name $\triangle$ by the equivalent name $\square$, and get

$$\square = ({}^{+}4 \times \square) + {}^{+}3.$$

We now have a familiar type of equation, which we can solve (for example) by using *transform operations*:

Subtract $\square$ from each side.

$$0 = ({}^{+}3 \times \square) + {}^{+}3$$

The truth set, evidently, is $\{{}^{-}1\}$.

But,

$$\triangle = \square,$$

so the *pair of numbers* is:

$$^{-}1 \rightarrow \square$$
$$^{-}1 \rightarrow \triangle.$$

Try to solve each pair.

(18) $\square + \triangle = 5$

$\square - \triangle = 3$

(18) $4 \rightarrow \square$

$1 \rightarrow \triangle$

This begins a sequence of questions which *may*(!) induce the children to discover the *add-or-subtract* method.

(19) $\square + \triangle = 14$

$\square - \triangle = 10$

(19) $12 \rightarrow \square$

$2 \rightarrow \triangle$

(20) $(2 \times \square) + (3 \times \triangle) = 16$

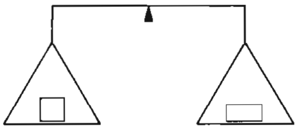
$(2 \times \square) - (3 \times \triangle) = 4$

(20) $5 \rightarrow \square$

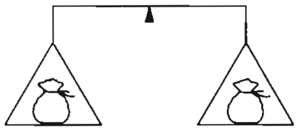
$2 \rightarrow \triangle$

(21) Jill has another method for solving simultaneous equations. She says her method depends upon this idea:

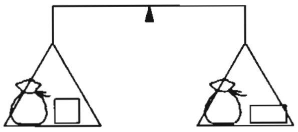
If these two boxes balance



and if these two bags balance,



will this balance, or not?



Can you guess Jill's method?

(21) Yes, it will balance. The description of Jill's method is left up to you.

[page 92]

Can you solve each pair of equations?

(22) $(2 \times \square) + (3 \times \triangle) = 16$

$(2 \times \square) - (3 \times \triangle) = 28$

(22) $\mathbf{11} \rightarrow \square$

$\mathbf{^-2} \rightarrow \triangle$

(23) $(5 \times \square) + (3 \times \triangle) = 38$

$(5 \times \square) + (2 \times \triangle) = 32$

(23) $\mathbf{4} \rightarrow \square$

$\mathbf{6} \rightarrow \triangle$

Questions 23 and 24 *may* (no guarantee!) lead the children to discover the idea of *subtracting one equation from the other.*

(24) $(7 \times \square) + (5 \times \triangle) = 13$

$(7 \times \square) + \triangle = {^-3}$

(24) $\mathbf{4} \rightarrow \triangle$

$\mathbf{^-1} \rightarrow \square$

(25) $\triangle = 2 \times \square$

$(5 \times \triangle) + (3 \times \square) = 143$

(25) $\mathbf{11} \rightarrow \square$

$\mathbf{22} \rightarrow \triangle$

Substituting, we get

$$\left[5 \times (2 \times \square)\right] + (3 \times \square) = 143$$
$$(10 \times \square) + (3 \times \square) = 143$$
$$13 \times \square = 143$$

with the truth set $\{11\}$.

Question 25, plus (perhaps) others like it that you yourself make up, may lead to the idea of *solving* one equation and then *substituting* into the other.

This is a lesson where it is important to remember:

> It does not matter (very much) whether the children learn all these methods for solving simultaneous equations; it does not matter (too much) that the children solve *any* simultaneous equations; it *does* matter (very much!) that *whatever they do with this topic should be approached in a spirit of originality, cleverness, understanding, confidence, and fun!*

Here, are two horns of the dilemma. If the children discover several systematic methods for solving simultaneous equations, and have fun doing it, that is wonderful. If they get the concept of one pair of numbers that satisfies two equations, and can enjoy easy problems (using the method of guessing), that is very nice. If they are *pushed* through some dreary lessons on simultaneous equations, that will be a *big mistake*.

(26) $\triangle = (3 \times \square) + 2$

$\triangle = (4 \times \square) + {}^{-}5$

(26) **7** → $\square$

23 → $\triangle$

$$(3 \times \square) + 2 = (4 \times \square) + {}^{-}5$$
$$(3 \times \square) + 7 = 4 \times \square$$
$$7 = \square$$
$$\{7\}$$

(27) How many methods do you know for solving simultaneous equations?

(27) **This question is less important than the question: "Was it exciting?"**

chapter 46 / *Pages 92–93 of Student Discussion Guide*

MORE WORD PROBLEMS

The word problems in this chapter generally require two placeholders for two unknowns, and involve two or more simultaneous equations or inequalities.

Chapter 46
MORE WORD PROBLEMS

ANSWERS AND COMMENTS

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(1) If Henry were three years older, he would be twice as old as Eddy. If Henry were two years younger, he and Eddy would be the same age. How old is each boy?

(1) **Eddy, 5**
Henry, 7

We translate the "English" into "algebra":

Henry's age	H
Eddy's age	E
Twice Eddy's age Twice as old as Eddy	$2 \times E$
If Henry were three years older, his age would be	$H + 3$
If Henry were three years older, he would be twice as old as Eddy.	$H + 3 = 2 \times E$
If Henry were two years younger, his age would be	$H - 2$
Henry and Eddy would be the same age	$H - 2 = E$

Consequently, we have the system of two simultaneous linear algebraic equations:

$$\begin{cases} H + 3 = 2 \times E \\ H - 2 = E \end{cases}$$

The roots of this system are: $H = 7$, $E = 5$.

(2) Four years ago Al was twice as old as Gene. Furthermore, $5\frac{1}{4}$ years ago Al was three times as old as Gene. How old is Al today?

(2) **Al, 9**
Gene, $6\frac{1}{2}$

Al's age today	A
Gene's age today	G
Al's age four years ago	$A - 4$
Gene's age four years ago	$G - 4$
Four years ago Al was twice as old as Gene	$A - 4 = 2 \times (G - 4)$

Similarly, for the second equation, we get

$$A - 5\tfrac{1}{4} = 3 \times (G - 5\tfrac{1}{4}).$$

We get the system:

$$\begin{cases} A - 4 = 2 \times (G - 4) \\ A - 5\tfrac{1}{4} = 3 \times (G - 5\tfrac{1}{4}) \end{cases}$$

We can simplify this by rewriting:

$$\begin{cases} A = (2 \times G) - 4 \\ A = (3 \times G) - 10\tfrac{1}{2} \end{cases}$$

Evidently, the roots are: $G = 6\frac{1}{2}$, $A = 9$.

(3) I am thinking of two numbers. If you add them together, you get 31; and if you subtract the second number from the first, you get three. Can you tell what the numbers are?

(3) **17 and 14**

Say the two numbers are $\square$ and $\triangle$:

$$\begin{cases} \square + \triangle = 31 \\ \square - \triangle = 3 \end{cases}$$

The roots are: $17 \rightarrow \square$, $14 \rightarrow \triangle$.

(4) John had a dream about airplane pilots and sports cars. When all the airplane pilots in John's dream got into sports cars, there was one pilot per car, and seven cars were still empty. Then one half of the pilots got into a big spaceship and took off for Venus. After that, when all of the remaining pilots got into the sports cars, there was one pilot per car, but 29 cars were empty.

How many sports cars were there in John's dream?

(4) **51**

Number of pilots	P
Number of sports cars	S

$$\begin{cases} P = S - 7 \\ \tfrac{1}{2} \times P = S - 29 \end{cases}$$

Evidently, $P = 44$, $S = 51$.

(5) Before Christmas, every girl in Miss Wilson's class had 3 scarves, except for Cathy, who had

[page 93]

3 orange and 2 black scarves. On Christmas Day, one half of the girls each received a new scarf. After Christmas, the class had 16 per cent more scarves than they had before.

How many girls are there in Miss Wilson's class?

(5) **16 girls**

We can attack this problem as follows. Since we do not know, *a priori*, how many girls there are in Miss Wilson's class, we can represent this number by a variable, N (say), and write some algebraic expressions (actually "functions") involving this variable:

Before Christmas, $N - 1$ girls had 3 scarves each, for a total of $3 \times (N - 1)$ scarves.

In addition, Cathy had 5 scarves.

Therefore, all N girls in Miss Wilson's class (including Cathy) had a grand total of $3 \times (N - 1) + 5$ scarves before Christmas.

This expression can be simplified (using the distributive law, laws for signed numbers, and commutative and associative laws for addition) to read

$$(3 \times N) + 2.$$

On Christmas day, $N/2$ of the girls received one new scarf each, so that the class had $N/2$ additional scarves. This increment represented a 16 percent increase:

$$\frac{N}{2} = \frac{16}{100} \times [(3 \times N) + 2].$$

This equation can be simplified by a sequence of transform operations:

$$N = \tfrac{16}{50} \times [(3 \times N) + 2]$$

$$N = \tfrac{8}{25} \times (3 \times N) + \tfrac{16}{25}$$

$$\tfrac{25}{25}N - \tfrac{24}{25}N = \tfrac{16}{25}$$

$$N = 16.$$

Consequently, we see that this tale cannot be true unless there are 16 girls in Miss Wilson's class.

(6) In a certain experiment, some protons collide with some electrons, and there are the same number of protons as there are electrons.

When the experiment is repeated with twice as many protons and three times as many electrons, there are 25 electrons left over.

How many protons were used in the original experiment?

(6) **25**

Numbers of protons in first experiment P
Number of electrons in the first experiment E

$$\begin{cases} P = E \\ (2 \times P) = (3 \times E) - 25 \end{cases}$$

Evidently, $E = 25$, $P = 25$.

(7) At Andy's candy store, chocolate nut bars cost 5 cents each, and imported Swedish sour balls cost 2 cents each. Jerry spent 31 cents, and he bought 11 pieces of candy. How many pieces of each kind of candy did he buy?

(7) **3 nutbars**
8 sourballs

Number of nut bars Jerry bought N
Number of sourballs Jerry bought S

$$\begin{cases} N + S = 11 \\ (5 \times N) + (2 \times S) = 31 \end{cases}$$

There are, of course, many things we might do at this point. Suppose we proceed like this:

$$\begin{cases} (2 \times N) + (2 \times S) = 22 \\ (5 \times N) + (2 \times S) = 31 \end{cases}$$

Evidently, $(3 \times N)$ must equal 9 (cf. the two equations).

(8) Jerry figured out that during all of last year, he spent $33.39 at Andy's candy store, and he bought (and ate) a total of 1209 pieces of candy.

How many pieces of each kind of candy did he buy?

(8) **307 nut bars**
902 sour balls

Similar to problem 7 above, we get

$$\begin{cases} N + S = 1209 \\ (5 \times N) + (2 \times S) = 3339 \end{cases}$$

Simplifying,

$$\begin{cases} (2 \times N) + (2 \times S) = 2418 \\ (5 \times N) + (2 \times S) = 3339 \end{cases}$$

$$\therefore\ 3 \times N = 921$$

$$N = 307$$

$$S = 1209 - 307 = 902$$

(9) I am thinking of two positive integers. If you square the first and add it to the second, you get 16. If you square the second and add it to the first, you get 52. What two numbers am I thinking of?

(9) **3 and 7**

First positive integer F
Second positive integer S

We know:

$$\begin{cases} 0 < F \\ 0 < S \\ F^2 + S = 16 \\ S^2 + F = 52 \\ F, S \text{ both belong to the set of integers} \end{cases}$$

Now $F^2 + S = 16$ and $0 < S$ together imply $F^2 < 16$. Since F is a positive integer, it must be either 1, 2, or 3.

We can try this with the roles of F and S reversed: $S^2 + F = 52$ and $0 < F$ together imply $S^2 < 52$, and since S is a positive integer, it must be an element of the set $\{1, 2, 3, 4, 5, 6, 7\}$.

Now comparing possibilities: if $F = 1$, then $F^2 = 1$, and $F^2 + S = 16$ implies $S = 15$.

We know, however, that S cannot be larger than 7, so this is impossible.

At this stage, we know that F is an element of the set $\{2, 3\}$ and S is an element of the set $\{1, 2, 3, 4, 5, 6, 7\}$.

Suppose $S \leq 6$; then $S^2 \leq 36$, and $S^2 + F = 52$ is impossible. Hence, $S > 6$, i.e., $S = 7$.

We now know definitely that $S = 7$. Therefore, $S^2 + F = 52$ implies $49 + F = 52$ and $F = 3$.

Notice that *these are the only two possible numbers*, as the preceding logic shows.

(10) Representatives of Pennsylvania, Ohio, Massachusetts, Vermont, and California met in Albany, New York, to buy the drinking-water rights to the Erie Canal.

The California man thought he knew how much money he would need to close a quick deal, so he sent home a code telegram:

SEND.

It turned out this was not enough money, so he sent home a telegram asking for an additional amount:

MORE.

Then he was afraid that one or the other of these telegrams might have been intercepted in Colorado, so he sent a third code telegram asking for the total:

MONEY.

The amount he requested arrived, and he bought the rights just before the man from Vermont was able to arrange a trade involving three mountains that were to be moved to Oswego for winter skiing.

How much did California pay for the drinking-water rights to the Erie Canal?

(10)

The main point here is to show that these are the only possible answers. If you use various kinds of considerations on the various possibilities, it is possible to prove the following statements: M must represent the number 1. S is an element of the set that contains 8 and 9. If S represents 8, then O represents 0, which is impossible. Therefore, S represents 9. O is an element of the set that contains 0 and 1. Since O cannot represent the number 1, it must represent the number 0. R represents the number 8. $D + E = Y + 10$. 12 is less than or equal to $D + E$. Since E does not represent the numbers 2, 3, or 4, it must be an element of the set that contains 5 and 6. E does not represent the number 6. Therefore, the letter E represents the number 5. $E + 1 = N$. Therefore, N represents the number 6. D represents the number 7. Y represents the number 2.

$$\begin{array}{r} \mathbf{SEND} \\ \mathbf{9567} \\ \mathbf{MORE} \\ \underline{\mathbf{1085}} \\ \mathbf{MONEY} \\ \mathbf{10652} \end{array}$$

(11) A farmer has □ hens and △ rabbits. These animals (the hens and rabbits, we don't count the farmer) have 50 heads and 140 feet. What else do you know?

(11) **30 hens***

20 rabbits

---

* This problem is discussed in a delightful fashion by Professor George Polya in his book, *Mathematical Discovery in Understanding, Learning, and Teaching Problem-Solving* (John Wiley and Sons, Inc., New York, N. Y., 1962).

A short form of this discussion goes like this:

$$\left\{\begin{array}{rl} \square + \triangle = 50 & \text{(counting heads)} \\ (2 \times \square) + (4 \times \triangle) = 140 & \text{(counting feet)} \end{array}\right.$$

If all hens will please stand (for a brief moment) on one leg, and all rabbits will please stand on their hind legs only, then (for this brief moment) there are this many feet on the ground:

$$\square + (2 \times \triangle) = 70.$$

Hence we have the system:

$$\left\{\begin{array}{r} \square + \triangle = 50 \\ \square + (2 \times \triangle) = 70 \end{array}\right.$$

Therefore, $20 \rightarrow \triangle$ and $30 \rightarrow \square$.

chapter 47 / Pages 94–96 of Student Discussion Guide

TRUTH SETS

This chapter is the type usually labeled "review." But do not breath a word of this to the children! Fifth and sixth graders are usually so eager to learn new things that it aggravates them to hear, "today we will have a review lesson."

Chapter 47
TRUTH SETS

[page 94]

Can you find the truth set for each open sentence?

(1) $(\square \times \square) - (8 \times \square) + 15 = 0$ $\{\ ,\ \}$

(2) $(\square \times \square) - (15 \times \square) + 36 = 0$ $\{\ ,\ \}$

(3) $(\square \times \square) - ({}^{+}3 \times \square) + {}^{-}10 = 0$ $\{\ ,\ \}$

(4) $(\square \times \square) - ({}^{+}9 \times \square) + {}^{-}22 = 0$ $\{\ ,\ \}$

(5) $(\square \times \square) - ({}^{-}7 \times \square) + {}^{-}18 = 0$ $\{\ ,\ \}$

(6) $(\square \times \square) - ({}^{-}5 \times \square) + {}^{+}6 = 0$ $\{\ ,\ \}$

(7) $(\square \times \square) - (15 \times \square) + 81 = 31$ $\{\ ,\ \}$

(8) $(\square \times \square) - (12 \times \square) + 32 = 5$ $\{\ ,\ \}$

(9) $(\square \times \square) + 96 = (35 \times \square)$

ANSWERS AND COMMENTS

(1) **{5, 3}**

(2) **{12, 3}**

(3) **{$^{+}$5, $^{-}$2}**

(4) **{$^{+}$11, $^{-}$2}**

(5) **{$^{-}$9, $^{+}$2}**

(6) **{$^{-}$2, $^{-}$3}**

(7) **{5, 10}**

This equation is not in our standard form; the *two secrets* will not work until the equation is rewritten so as to get a zero on the right-hand side. It can be rewritten by using the transform operation: "subtract 31 from each side."

(8) **{3, 9}**

See the discussion for question 7.

(9) **{32, 3}**

Subtracting 35 × $\square$ from both sides of the equation gives

$$(\square \times \square) + 96 - (35 \times \square) = (35 \times \square) - (35 \times \square).$$

Since $(35 \times \square) - (35 \times \square)$ is actually zero, we have:

$$(\square \times \square) + 96 - (35 \times \square) = 0.$$

It is necessary to rearrange the order of the terms on the left-hand side; the "changing signs" axiom and the axioms on commutative and associative laws for addition are used to do this one step at a time. However, instead of doing this one step at a time, with complete detail, you might simply write:

$$(\square \times \square) - (35 \times \square) + 96 = 0.$$

This is now in standard form, and we can look for numbers whose product is 96, and whose sum is 35. The truth set, evidently, is {32, 3}.

(10) Jerry says that he used a transform operation on

$$(\square \times \square) + 96 = (35 \times \square)$$

before he solved it.
What transform operation did Jerry use?

(10) **He subtracted $(35 \times \square)$ from each side.**

Actually, he may have followed this up by using a sequence of identities, as follows:

$$
\begin{aligned}
&[(\square \times \square) + 96] - (35 \times \square) \\
&\qquad = (35 \times \square) - (35 \times \square) && \text{CS}\\
&[(\square \times \square) + 96] + {}^{\circ}(35 \times \square) \\
&\qquad = (35 \times \square) + {}^{\circ}(35 \times \square) && \text{L Opp}\\
&[(\square \times \square) + 96] + {}^{\circ}(35 \times \square) = 0 && \text{ALA}\\
&(\square \times \square) + [96 + {}^{\circ}(35 \times \square)] = 0 && \text{CLA}\\
&(\square \times \square) + [{}^{\circ}(35 \times \square) + 96] = 0 && \text{ALA}\\
&[(\square \times \square) + {}^{\circ}(35 \times \square)] + 96 = 0 && \text{CS}\\
&(\square \times \square) - (35 \times \square) + 96 = 0
\end{aligned}
$$

(See the answer to question 9, where, as an alternative, it is suggested that you skip over these details and go directly to the last line.)

(11) Is this an identity?

$$A + (B \times C) = A - ({}^{\circ}B \times C)$$

(11) **Yes**

(12) Can you make a derivation for

$$A + (B \times C) = A - ({}^{\circ}B \times C)?$$

(12) **If you have previously proved the theorem that**

$$^{\circ}(B \times C) = {}^{\circ}B \times C,$$

it is suggested that you use that theorem in this present proof.

(13) Is this an identity?

$$\square - \triangle = \square + {}^{\circ}\triangle$$

(13) **Yes; it is an axiom known as "changing signs."**

Can you make a derivation for each identity?

(14) $\square - \triangle = \square + {}^{\circ}\triangle$

(14) **No. It is an axiom, not a theorem.**

(15) $\square + \square = 2 \times \square$	(15) **Yes. This has been done earlier in this book. See Chapter 32.***
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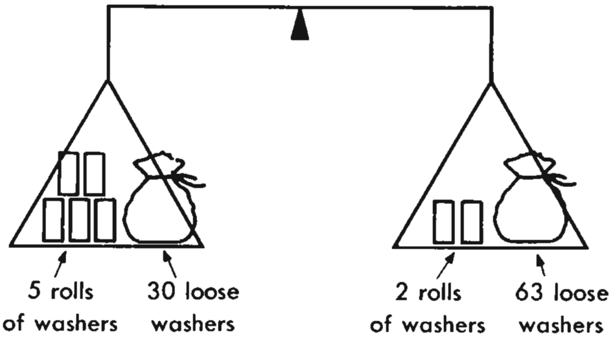
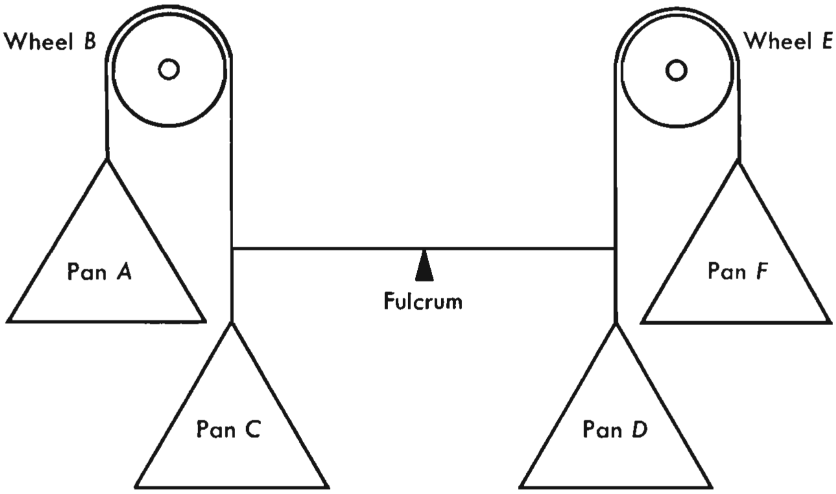
(16) $\square + \square + \square = 3 \times \square$	(16) **This is generally similar to question 15.**
(17) $(A + B) \times (A + B) = (A \times A) + (A \times B) + (B \times B)$	(17) **No. It is *not* an identity!**
Can you find the truth set for each quadratic equation?	
(18) $(\square \times \square) + (5 \times \square) + 6 = 0$ { , }	(18) $\{\mathbf{2, 3}\}$
(19) $(\square \times \square) = (24 \times \square) - 44$ { , }	(19) $\{\mathbf{22, 2}\}$ Rewrite as $(\square \times \square) - (24 \times \square) + 44 = 0.$
(20) $(\square \times \square) - (10 \times \square) + 21 = 0$ { , }	(20) $\{\mathbf{7, 3}\}$
(21) $(\square \times \square) - ({}^-10 \times \square) + {}^+21 = 0$ { , }	(21) $\{\mathbf{{}^-7, {}^-3}\}$
Can you solve these equations?	
(22) $(5 \times \square) + 30 = (2 \times \square) + 63$	(22) $\{\mathbf{11}\}$
(23) $(5 \times \square) + 11 = 46 - (2 \times \square)$	(23) $\{\mathbf{5}\}$ You might want to use transform operations to write this as: $(5 \times \square) + 11 + (2 \times \square) = 46$ $(7 \times \square) + 11 = 46$ $(7 \times \square) = 35$ The truth set, evidently, is: $\{5\}$.
Can you make a balance picture for each open sentence?	
(24) $(5 \times \square) + 30 = (2 \times \square) + 63$	(24) 
(25) $(5 \times \square) + 11 = 46 - (2 \times \square)$	(25) **This balance picture uses an ingenious device of Warwick Sawyer (Professor of Mathematics, Wesleyan University):**

---

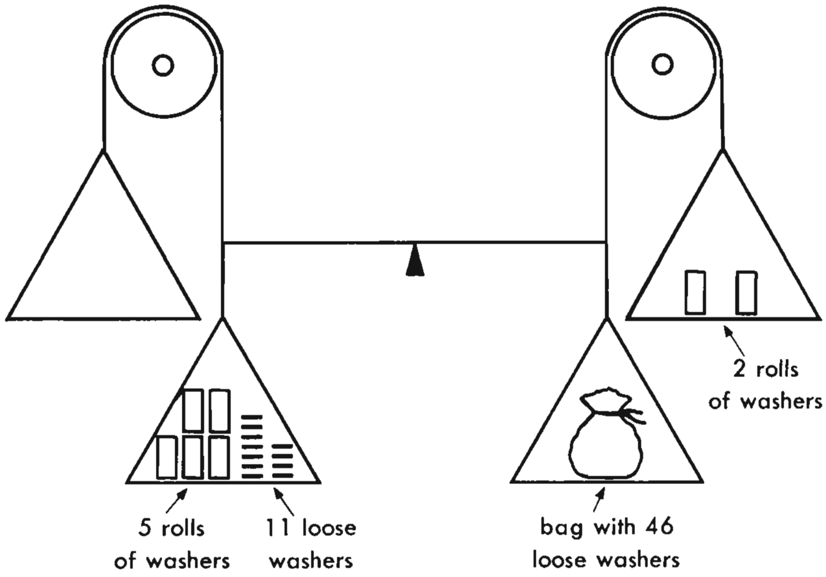
* Two different derivations are handled very nicely by a fifth-grade class on tape number D-1.



The wheels *B* and *E* are attached to a supporting stand in order to provide a pulley arrangement for pans *A* and *F*.

Notice that placing a washer on each pan has an effect as follows:

Pan	**Physical effect**	**Mathematical equivalent**
A	**raises left-hand side**	**subtraction from left-hand side of equation**
C	**weighs down left-hand side**	**addition to left-hand side of equation**
D	**weighs down right-hand side**	**addition to right-hand side of equation**
F	**raises right-hand side of balance**	**subtraction from right-hand side of equation**



If these Sawyer-type balance pictures appeal to you, you (and your children) may enjoy working with them somewhat systematically.

Can you find the truth set for each open sentence?

(26) $(\square \times \square) + ({}^{+}15 \times \square) + {}^{+}50 = 0$ $\{\quad , \quad\}$

(27) $(\square \times \square) - ({}^{+}24 \times \square) + {}^{+}63 = 0$ $\{\quad , \quad\}$

(26) $\{{}^{+}\mathbf{10},\ {}^{+}\mathbf{5}\}$

(27) $\{{}^{+}\mathbf{21},\ {}^{+}\mathbf{3}\}$

(28) $\square + \square + (3 \times \square) - 10 = 10$ $\{\quad\}$	(28) $\{4\}$ This is equivalent to: $(5 \times \square) - 10 = 10$ $(5 \times \square) = 20$
(29) $18 + {}^{\circ}\square = 0$ $\{\quad\}$	(29) $\{18\}$ **Notice that** ${}^{\circ}\boxed{18} = {}^{-}18$.
(30) $\square + \square = 7$ $\{\quad\}$	(30) $\{3\frac{1}{2}\}$
(31) $(\square \times \square) - ({}^{-}22 \times \square) + {}^{+}40 = 0$ $\{\quad,\quad\}$	(31) $\{{}^{-}20, {}^{-}2\}$
[page 96] (32) $(\square \times \square) + ({}^{-}6 \times \square) + {}^{-}27 = 0$ $\{\quad,\quad\}$	(32) $\{{}^{-}9, {}^{+}3\}$
(33) $(\square \times \square) + ({}^{+}15 \times \square) + {}^{-}100 = 0$ $\{\quad,\quad\}$	(33) $\{{}^{+}20, {}^{-}5\}$
(34) $(\square \times \square) - ({}^{-}15 \times \square) + {}^{-}100 = 0$ $\{\quad,\quad\}$	(34) $\{{}^{-}20, {}^{+}5\}$

Can you make a "machine" (or formula) that will tell the truth set for each equation?

(35) $a + \square = b$	(35) $\square = b - a$
(36) $(a \times \square) = b$	(36) $\square = \frac{b}{a}$
(37) $(a \times \square) + b = c$	(37) $\square = \frac{c - b}{a}$
(38) $(\square \times a) + (\square \times b) = c$	(38) $\square = \frac{c}{a + b}$
(39) $(\square \times e) + (\square \times f) = w$	(39) $\square = \frac{w}{e + f}$
(40) $(\square \times a) + (\square \times 4) = h$	(40) $\square = \frac{h}{a + 4}$
(41) $(\square \times a) + \square = c$	(41) $\square = \frac{c}{a + 1}$
(42) Alice used two identities to rewrite the equation $(\square \times a) + \square = c$ before she made up a machine to solve it. What two identities did Alice use?	(42) $(\square \times a) + (\square \times 1) = c$ **L1** $\square \times (a + 1) = c$ **DL**

(43) Can you solve this equation?

$$\square \times (\square - 12) = {}^{-}35$$

(43) $\{{}^{+}5, {}^{+}7\}$

$$\square \times (\square - 12) = {}^{-}35$$

CS

$$\square \times (\square + {}^{-}12) = {}^{-}35$$

DL

$$(\square \times \square) + (\square \times {}^{-}12) = {}^{-}35$$

CLM

$$(\square \times \square) + ({}^{-}12 \times \square) = {}^{-}35$$

Theorem: $^{\circ}(A \times B) = ({}^{\circ}A) \times B$

$$(\square \times \square) + {}^{\circ}({}^{+}12 \times \square) = {}^{-}35$$

CS

$$(\square \times \square) - ({}^{+}12 \times \square) = {}^{-}35$$

Add $^{+}35$ to both sides (a "transform operation")

$$(\square \times \square) - ({}^{+}12 \times \square) + {}^{+}35 = 0$$

(44) Lex used some identities to rewrite the equation

$$\square \times (\square - 12) = {}^{-}35$$

before he solved it.
How did he do it?

(44) **See the answer to question 43.**

(45) Can you solve this equation?

$$\square \times (\square + 22) = 60 + (5 \times \square)$$

(45) $\{{}^{-}20, {}^{+}3\}$

(46) Can you rewrite the equation in problem 45 in order to make it look more familiar?

(46)

$$(\square \times \square) + (\square \times 22) = 60 + (5 \times \square)$$
$$(\square \times \square) + (22 \times \square) - (5 \times \square) + {}^{-}60 = 0$$
$$(\square \times \square) + (17 \times \square) + {}^{-}60 = 0$$
$$(\square \times \square) - ({}^{-}17 \times \square) + {}^{-}60 = 0$$

Can you solve each equation?

(47) $(7 \times \square) + 11 = (4 \times \square) + 56$

(47) $\{15\}$

Use balance pictures, if necessary.

(48) $(\square \times \square) - (242 \times \square) + 6360 = 0$

(48) $\{212, 30\}$

A systematic factorization into primes may be the best approach.

$$\begin{aligned} 6360 &= 636 \times 10 \\ &= 3 \times 212 \times 10 \\ &= 3 \times 2 \times 106 \times 10 \\ &= 3 \times 2 \times 2 \times 53 \times 10 \\ &= 3 \times 2 \times 2 \times 53 \times 2 \times 5 \\ &= 2^3 \times 3 \times 5 \times 53 \end{aligned}$$

Now separate these into two factors in a systematic way to see if any combination gives the sum 242:

1 × 6360	1 + 6360 = 6361	No
2 × 3180	2 + 3180 = 3182	No
3 × 2120		No
4 × —	You can skip some of the calculations . . .	No
5 × —		No
6 × —		No
7 × —		No
8 × 795	8 + 795	No
10 × 636	10 + 636	No
12 × 530		No
15 × 400		No
20 × 318		No
30 × 212	30 + 212 = 242	*Voilà!*

chapter 48 / *Pages 97–98 of Student Discussion Guide*

PRIMES

Chapter 48
PRIMES

ANSWERS AND COMMENTS

[page 97]

(1) Do you know what we mean by a **prime?**

(1) **A prime is a number with no factors (within the number system of whole numbers), except *itself* and one.**

Thus 7 is a prime, because 7 = 1 × 7, but there are no *other* whole number factorizations. Also, 8 is *not* a prime, because 8 = 4 × 2.

The following *are* primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, . . .

The following *are not* primes: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, . . .

A special ruling is made in the case of the whole number one. We agree to say that one is not *a prime. (This is a matter of convenience in certain work in number theory.) This special ruling is not really important, and if you feel it is an unpardonable disruption of orderly procedure, you may prefer to ignore it. It does not matter much.*

(2) Is 2 a prime?

(2) **Yes**

(3) Is 3 a prime?

(3) **Yes**

(4) Is 4 a prime?

(4) **No**

Four is not a prime, since 4 = 2 × 2.

(5) Can you name another number that is a prime?

(5) **See answer to question 1.**

(6) Can you name a number that is **not** a prime?

(6) **See answer to question 1.**

(7) Is 100 a prime?

(7) **No**

One hundred is not a prime because 100 = 10 × 10. (Or, for that matter,

$$\begin{aligned}100 &= 4 \times 25\\ &= 2 \times 50\\ &= 2 \times 2 \times 25\\ &= 2 \times 2 \times 5 \times 5.)\end{aligned}$$

(8) Is 5 a prime?

(8) **Yes**

(9) Is 6 a prime?

(9) **No**

Factoring 6 into *prime factors*, we get

$$6 = 2 \times 3.$$

(10) Is 7 a prime?

(10) **Yes**

(11) Is 8 a prime?

(11) No

Factoring 8 into *prime factors*, we get

$$8 = 2 \times 2 \times 2.$$

(12) Tony says this set contains only primes.

{2, 3, 7, 11, 21, 23, 13, 5, 41}

Do you agree?

(12) No. Twenty-one is not a prime, since 21 = 3 × 7.

Let us now see whether 41 is a prime. First, notice that $7 \times 7 = 49$. Therefore *if 41 does have any prime factors, at least one of them must lie in the interval* $2 \leq n < 7$.

This means we need look for factors only among the set

$$\{2, 3, 4, 5, 6\}.$$

We might be wise to borrow from number theory the symbol "|", meaning "divides." Thus, the following are all *true* statements:

$2 \mid 4$ "2 divides 4"
$3 \mid 27$ "3 divides 27"
$3 \mid 30$
$7 \mid 21$

whereas, the following are all *false* statements (since we are using only integers):

$2 \mid 5$ "2 divides 5"
$3 \mid 7$
$4 \mid 21$

We can now proceed in a systematic way:

$2 \mid 41$ False
$3 \mid 41$ False
$4 \mid 41$ Must be *false*, since $2 \mid 41$ was.
$5 \mid 41$ False
$6 \mid 41$ Must be false, since $6 \mid 41$ implies $2 \mid 41$ and $3 \mid 41$.

Consequently, 41 is a prime.

(13) Ellen says there are **no** primes in this set.

{4, 8, 12, 6, 18, 36, 87}

Do you agree?

(13) Yes. The only possible question concerns 87. However, $3 \mid 90$, and therefore $3 \mid 87$. Conclusion: Ellen is right.

(14) Marie says there are **no** primes in this set.

{16, 32, 212, 1066, 57, 31, 121, 99}

Do you agree?

(14) No; 31 is a prime.

Two divides 16, 32, 212, and 1066. What about 57? Well, $3 \mid 60$, and therefore $3 \mid 57$.

How about 31? If 31 has any prime factors other than 1 and itself, at least one of these must be less than 6 (since $6 \times 6 = 36$). We need to test 2, 3, 4, and 5. (Actually, we can omit 4, since it is not a prime, and $4 \mid 31$ would imply $2 \mid 31$.)

$2 \mid 31$ False
$3 \mid 31$ False
$5 \mid 31$ False

Conclusion: 31 is a prime; consequently, Marie is wrong.

(15) What do we mean by a **prime?**
Jerry says he solved this equation

$$(\square \times \square) - (10 \times \square) + 24 = 0$$

this way:

First, he found the prime factors of 24:

$$24 = 12 \times 2$$
$$24 = (3 \times 4) \times 2$$
$$24 = (3 \times 2 \times 2) \times 2$$

Second, he tried to combine these to find two numbers that would add together to give 10:

Smallest	All the rest	
2	$3 \times 2 \times 2 = 12$	
$12 + 2 = 14$		☹
Next smallest	All the rest	
3	$2 \times 2 \times 2 = 8$	
$3 + 8 = 11$		☹
Next smallest	All the rest	
$2 \times 2 = 4$	$3 \times 2 = 6$	
$4 + 6 = 10$		☺

[page 98]

Can you use Jerry's method to solve these equations?

(16) $(\square \times \square) - (17 \times \square) + 72 = 0$

(15) **See answer to question 1.**

(16) {**8, 9**}

Now for the payoff. The reason for introducing primes was that the device of *prime factorization* gives us a *systematic* and *methodical* way to solve quadratic equations.

Let's use Jerry's method on the equation

$$(\square \times \square) - (17 \times \square) + 72 = 0.$$

First, get a prime factorization of 72:

$$\begin{aligned} 72 &= 2 \times 36 \\ &= 2 \times 4 \times 9 \\ &= 2 \times 2 \times 2 \times 3 \times 3 \end{aligned}$$

Second, compare sums, to see if we can get a sum of 17:

Smallest	All the rest		
2	$2 \times 2 \times 3 \times 3 = 36$	$2 + 36 = 38$	No
Next smallest			
3	$8 \times 3 = 24$	$3 + 24 = 27$	No
Next smallest			
$2 \times 2 = 4$	18	$4 + 18 = 22$	No
Next smallest			
$2 \times 3 = 6$	12	$6 + 12 = 18$	No
Next smallest			
$2 \times 2 \times 2 = 8$	9	$8 + 9 = 17$	Yes

(Before you discard Jerry's method as too complicated, you may want to consider Leibniz, quoted by Polya: "A method of solution is perfect if we can foresee at the start, and even prove, that following the method we shall attain our aim." Jerry's method guarantees us that *we shall either find the truth set, or else we shall know quite certainly that there are no whole numbers that will yield a true statement for the open sentence in question.*)

(17) $(\square \times \square) - (20 \times \square) + 96 = 0$

(17) **{8, 12}**

We begin by getting a prime factorization of 96.

$$\begin{aligned} 96 &= 3 \times 32 \\ &= 3 \times 16 \times 2 \\ &= 3 \times 4 \times 4 \times 2 \\ &= 3 \times 2^5 \end{aligned}$$

Arranging factors in order of size, we get:

Smallest	All the rest		
1	96	1 + 96 = 97	No
2	48	2 + 48 = 50	No
3	32	3 + 32 = 35	No
4	24	4 + 24 = 28	No
6	16	6 + 16 = 22	No
8	12	8 + 12 = 20	Yes

(18) $(\square \times \square) - (192 \times \square) + 8192 = 0$

(18) **{128, 64}**

$$\begin{aligned} 8192 &= 2 \times 4096 \\ &= 2 \times 4 \times 1024 \\ &= 2 \times 4 \times 2 \times 512 \\ &= 2^4 \times 2 \times 256 \\ &= 2^5 \times 2 \times 128 \\ &= 64 \times 128 \end{aligned}$$

We may not need a prime factorization. Let's try these out:

$$64 + 128 = 192.$$

Hence, the truth set is {128, 64}.

Can you solve these equations?

(19) $(\square \times \square) - (70 \times \square) + 1029 = 0$

(20) $(\square \times \square) - (36 \times \square) + 288 = 0$

(21) $(\square \times \square) - (72 \times \square) + 1260 = 0$

(22) $(\square \times \square) - (177 \times \square) + 176 = 0$

(23) $(\square \times \square) - ({}^-12 \times \square) + {}^-45 = 0$

(24) $(\square \times \square) + (5 \times \square) + 6 = 0$

(19) through (24) **These are left to you (or to your bright students).**

chapter 49 / *Pages 98–99 of Student Discussion Guide*

A LAW OF PHYSICS

Professor Robert Karplus of the University of California at Berkeley taught this lesson to a Madison Project class. Professor Karplus is an exceptionally fine theoretical physicist, whose reputation as a research scientist is international. In recent years, perhaps because of his own children, he has become interested in elementary school children, and he and some colleagues have prepared some excellent science materials for grades kindergarten through nine. All the material in the present chapter was devised by Professor Karplus.*

The point of the chapter is to contrast the linear (or simple Hooke's law) elasticity of a bent wire coat hanger with the nonlinear elasticity of rubber bands.

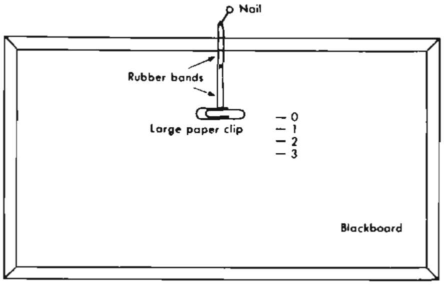
All of the materials used here are commonplace school items. For the rubber band balance, simply fashion a chain of rubber bands, by any convenient method, with a large paper clip at the lower end. For both balances, use either heavy metal washers or else some light paper pamphlets of equal weight for your *set of weights.*

Mark the blackboard to show the amount of stretching produced by *one* pamphlet suspended on the balance (i.e., clasped in the large paper clip), the amount of stretching produced by *two* pamphlets held by the paper clip, then by *three,* and so on. These marks will *not* be evenly spaced for the rubber-band balance.

Chapter 49
A LAW OF PHYSICS

[page 98]

Jerry wants to weigh some bags of washers. He is trying to make a balance like this.



(1) How can he decide where to put the marks, 0, 1, 2, 3, and so on?

(2) Tony says that these marks should be equally spaced. Do you agree?

ANSWERS AND COMMENTS

(1) By trial, as mentioned above.

(2) They will not turn out to be, provided the weight of each pamphlet is about the same. (The reason is that the rubber bands will *not* satisfy a linear Hooke's law relation.)

---

* The material and ideas produced by Professor Karplus and his colleagues are available from: Professor Robert Karplus, Physics Department, University of California, Berkeley, California.

(3) Can you fill in this table? [page 99]

□ is a placeholder for the number of washers.

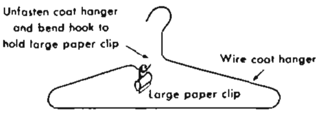
□	△
0	0
1	
2	
3	
4	
5	
6	
7	

△ is a placeholder for the distance that the rubber bands stretch.

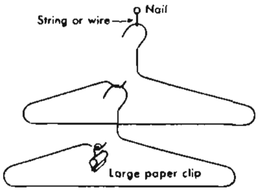
(4) Can you represent these numbers on a graph?

(5) Ellen says that a famous physicist named Hooke described how things stretch. Do you know what Hooke said?

(6) Sarah wants to make a balance also, but she wants to use two wire coat hangers instead of rubber bands:



You can combine two coat hangers to make an even better balance:



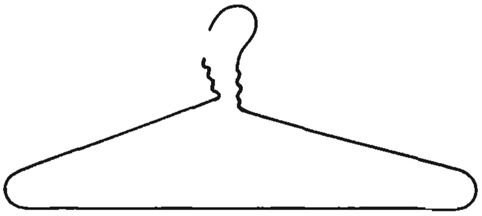
Can you finish Sarah's balance?

(3) You can measure this on your blackboard, in inches or in centimeters.

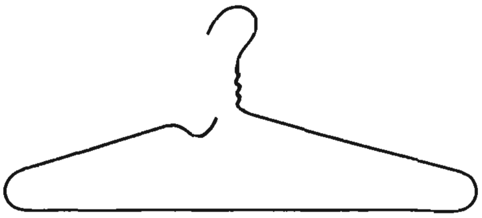
(4) You will need to use the data from your own class. The resulting graph should *not* turn out to be a straight line.

(5) Hooke said that a graph of force versus amount of stretching (such as you have made here) will sometimes be a straight line, and sometimes not. Which it is will depend upon the material being stretched (in this case, the rubber), and also upon how hard we are pulling (in this case, the weight of the washers or booklets being weighed).

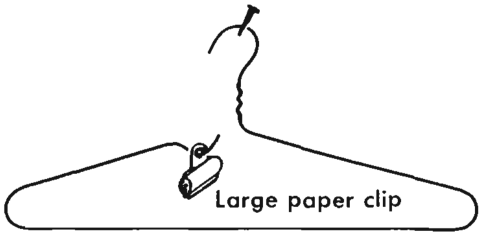
(6) For Sarah's balance, take two ordinary wire coat hangers and unwind them at the neck so that they look like this (more-or-less),



and then bend the new end to form a small hook.

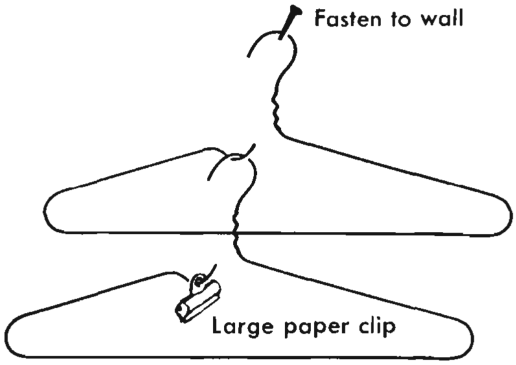


If you now suspend the large hook from a nail in the wall and hang a large paper clip onto the small hook,



you have a balance capable of weighing heavy metal washers or light paper pamphlets or notebooks or quiz books of (roughly) equal weight.

You can make an even better balance by using *two* hangers in *series*, like this:



(7) Can you make a table for the number of washers (□) and the distance that the clip moves (△), for Sarah's balance?

(8) Can you represent these numbers on a graph?

(9) Can you write the equation for this?

(10) Should the numbers "0", "1", "2", . . . on Sarah's balance be equally spaced?

(7) through (10) **Use data from your class. This time, the marks on the board should turn out to be equally spaced, and the graph should be a straight line.**

(11) Can you state Hooke's law?

(11) **Hooke's law, which was discovered by Robert Hooke, states that under certain circumstances (depending upon the nature of the materials, among other things) the numbers will be equally spaced. That is to say, the amount of stretching will be proportional to the weight hanging on the spring. From the work done in the present chapter, we have seen that this evidently does hold for coat hangers—at least to the degree of accuracy that we were able to tell from relatively crude measurements—and that it does not hold for rubber bands. You can read more about Hooke's law in any good college physics book.**

chapter 50 / *Pages 100–101 of Student Discussion Guide*

MACHINES IN GEOMETRY

As in earlier chapters, the word *machine* is used to mean *formula*. The reason for this use of a new word, which you may avoid if you wish, is our experience that college freshmen and other students do not usually understand the concept of the *general form* of a problem with parameters or variable coefficients or whatever and the possibility of solving a whole *class* of problems at one fell swoop.

For example, given a quadratic equation, such as $7x^2 + 35x - 28 = 0$, we can solve it by "completing the square" or by some other method. We have thus solved *one* problem.

But if we can find a notation to represent *all possible quadratic equations* (and we can, namely, $ax^2 + bx + c = 0$, $a \neq 0$), and if we can then solve this "equation in general form" (and we can, getting

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as a result), then we have just solved *infinitely many problems* at one blow. In fact, we have just solved *every possible quadratic equation!*

We wanted some terminology to dramatize this "killing infinitely many birds with one stone" idea, and so we introduced the word "machine."

This is what a machine (i.e., a *formula*) will do for us:

We can solve

$$\begin{aligned} 3 \times \Box &= 21 \\ 8 \times \Box &= 16 \\ 2 \times \Box &= 18 \\ 5 \times \Box &= 35 \\ 11 \times \Box &= 143 \end{aligned}$$

and *every other equation* of this same form, by writing the *general* equation

$$a \times \Box = b, \qquad a \neq 0$$

and solving it with the formula (or machine)

$$\Box = \frac{b}{a}.$$

That is, $\Box = \frac{b}{a}$ is the machine that will indicate the roots of *all* equations of the type $a \times \Box = b$, $a \neq 0$.

Chapter 50
MACHINES IN GEOMETRY

[page 100]

(1) Can you find the truth set for this open sentence?

$$3 \times \square = 21$$

(2) Make up an open sentence like this:

$$\underline{\quad} \times \square = \underline{\quad}$$

↑ Some number here ↑ Some number here

See if the other students can solve your equation.

Can you make up a "machine" to solve each equation?

(3) $a \times \square = b$

(4) $(a \times \square) + b = c$

(5) $(\square \times a) + (\square \times b) = c$

(6) Do you know what mathematicians call "machines"? (They have a different word.)

(7) If a rectangle is 3 inches wide, and 5 inches long, how many 1-inch squares can we fit into its **area?**

Can you show this in a picture?

(8) Do you know what we mean by **perimeter?**

ANSWERS AND COMMENTS

(1) $\{7\}$

(2) **There are many possible answers.**

(3) $\square = \dfrac{b}{a}$

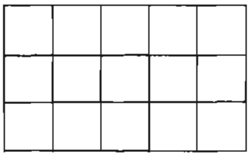
(4) $\square = \dfrac{c - b}{a}$

(5) $\square = \dfrac{c}{b + a}$

This one is tricky. After a large number of guesses, most classes come up with the correct machine. Of course, you may want to suggest rewriting the original equation, using the distributive law.

(6) **Formulas**

(7)



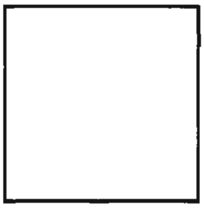
The rectangle contains 15 one-inch square units of *area*.

(8) **This is left to you to explain in your own way.**

It is sometimes helpful to begin by stressing the idea of a unit of *length*,

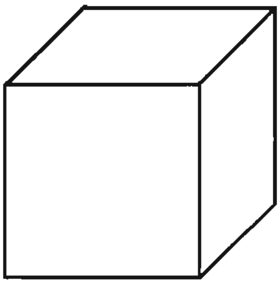
one inch—a unit of length

versus a unit of *area*,



one square inch—a unit of area

versus a unit of *volume*.



one cubic inch—a unit of volume

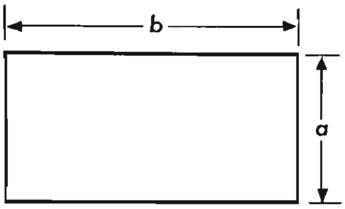
It is also often fun to introduce the units of length, area, and volume in the metric system by means of centimeter rods.*

(9) Can you find the perimeter of the rectangle in question 7?

(9) **16 inches. (Note that we use a unit of *length*.)**

(10) Can you make a machine that will give the **area** of **any** rectangle? (You will need to put letters as names for some of its measurements. Where?)

(10)



$A = a \times b$

(11) Try out your machine on a few rectangles. Does it work?

(11) **This is usually fun.**

(12) Can you make a machine that will give the **perimeter** of **any** rectangle?

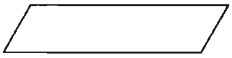
(12) **$p = (2 \times a) + (2 \times b)$ is one machine.**

There are others, such as

$$p = (a + b) + (a + b)$$
$$p = 2 \times (a + b),$$

and so on. It is an interesting algebra problem to see if these are "really the same."

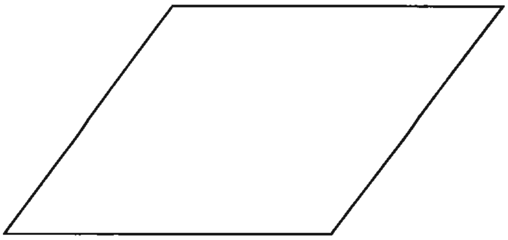
(13) Do you know the name for a figure shaped like this?



(13) **Parallelogram**

[page 101]

(14) See if you can find the **area** and the **perimeter** of this figure.



(On the student page this figure has dimensions of 5 inches by 6 inches.)

(14) **Area is 24 square inches; perimeter is 22 inches.**

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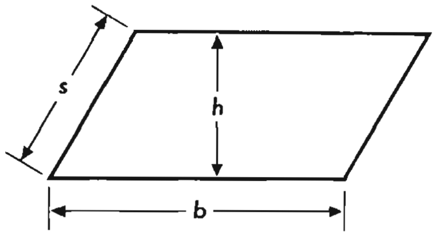
*A tape-recorded lesson showing this approach is available. For more information write to Robert B. Davis, Curriculum Laboratory, University of Illinois, Urbana, Ill. 61801.

(15) Can you use letters to indicate the important measurements of a parallelogram?

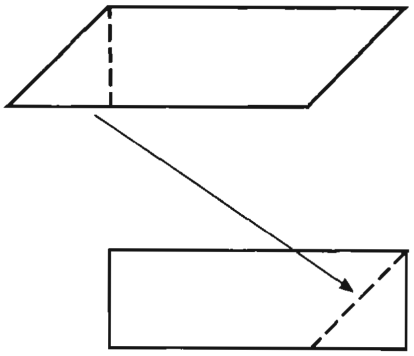
(15) **The *choice* of letters is up to the students, but here is *where* they must *go*:**



(16) Can you make up a machine to give the **area** of **any** parallelogram?

(16) $\mathbf{A = b \times h}$

The secret is to cut off a triangular piece from one end, and move it over to the other end:



The area of the resulting rectangle (using the letters of question 15) is evidently $A = b \times h$. This, then, is evidently *also* the area of the original *parallelogram.*

(17) Can you make up a machine that will give the **perimeter** of **any** parallelogram?

(17) $p = (b + s) + (b + s)$

or

$p = (2 \times b) + (2 \times s)$

or

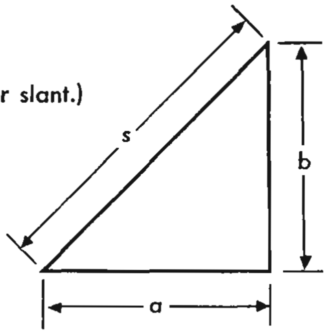
$p = 2 \times (b + s)$

etc.

You may (if you wish) invoke the distributive law, etc., to show the equivalence of these various formulas.

(18) Can you use letters to show the important measurements of a **right triangle?**

(18) **The choice of letters is up to the students, but here is where they must go:**



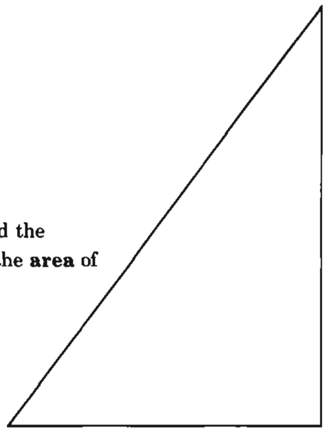
(The letter *s* is for slant.)

(19) Can you make up a machine that will give the **perimeter** of **any** right triangle?

(19) **Using letters of question 8, here is a formula for perimeter:**

$$p = a + b + s.$$

(20) Can you make up a machine that will give the **area** of **any** right triangle?



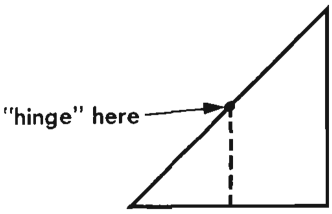
(21) Can you find the **perimeter** and the **area** of this triangle?

(On the student page this figure has dimensions: $a = 3$ inches, $b = 4$ inches, $s = 5$ inches.)

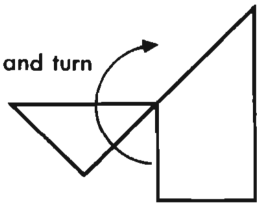
(20) $\boldsymbol{A = \frac{1}{2} \times (a \times b)}$

This can be done by cutting a piece and relocating it:

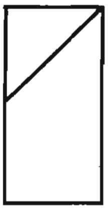
"hinge" here



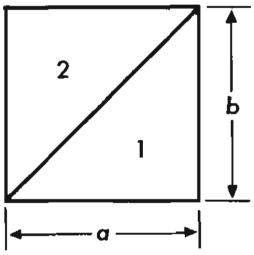
and turn



to get this:



A more common method is to fit together two triangles



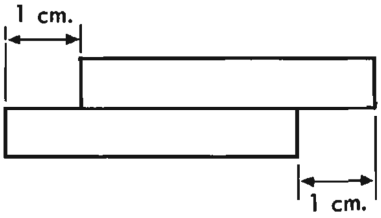
so that the area of the rectangle is A (rectangle) $= a \times b$. Since the triangle is half of this,

$$A \text{ (triangle)} = \tfrac{1}{2} \times (a \times b).$$

(21) **Perimeter is 12 inches;**
area is 6 square inches.

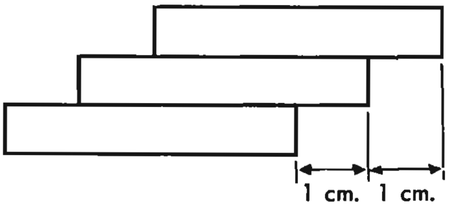
A very fine problem, in the spirit of this chapter, has been devised by Professor David Page, Director of the University of Illinois "Arithmetic Project." Here it is in several stages using a rod made from a block of wood—a parallelepiped of the dimensions 1 × 1 × 7 centimeters:

(a) Hold two of the block rods as shown:



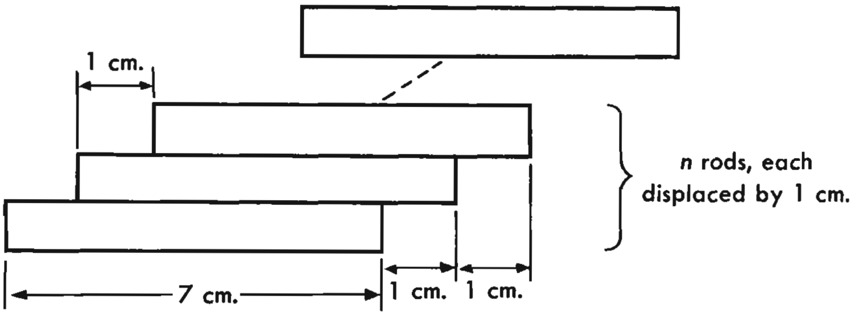
Ask the class to calculate the volume and exposed surface area.

(b) Now repeat, with three of the block rods (each is, of course, $1 \times 1 \times 7$ cm.):



(c) Now, similarly, compute volume and exposed surface area for a "stairway" of 4 rods, then 5 rods, etc.

(d) Here is where the real fun comes! Ask the children to find the volume and the surface area of a "stairway" of n rods:



Professor Page has predicted the outcome; this has been tried with various Madison Project classes, and Page's predictions are born out with uncanny accuracy:

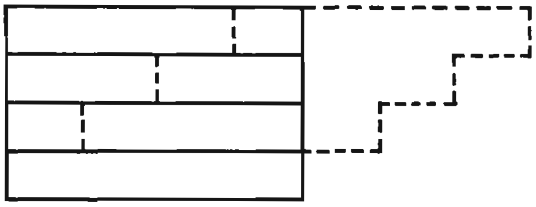
(a) Some children say:

Front has area of $7 \times n$ square centimeters;
back has area of $7 \times n$ square centimeters;
top has area of 7 square centimeters;
bottom has area of 7 square centimeters;
left end (vertical part) has area of n square centimeters;
left end (part facing upward has area of $n - 1$ square centimeters;
similarly, right ends have area of $n + (n - 1)$ square centimeters; therefore
total surface area:

$$A = (14 \times n) + 14 + (2 \times n) + 2 \times (n - 1)$$

(b) Some children say:

Slide "stairway" into a parallelepiped:



Surface area of "block" is $(14 \times n) + (2 \times n) + 14$.

Now! Slide $(n - 1)$ upper rods into original oblique position, i.e., 1 centimeter to right with respect to rod just below.

As you slide each rod, *2 square centimeters of additional* area are exposed. You slide $(n - 1)$ rods. Therefore, we must add $2 \times (n - 1)$, to get the final result:

$$A = (14 \times n) + (2 \times n) + 14 + 2 \times (n - 1).$$

(c) Some children say:

Two "outside" rods have $21 + 2 + 1 = 24$ square centimeters of exposed surface; $n - 2$ "inside" rods have $14 + 2 + 2 = 18$ square centimeters of exposed surface. Therefore, total surface: $A = (2 \times 24) + (n - 2) \times 18$.

(d) Some children say:

One rod has $(4 \times 7) + 2 = 30$ square centimeters of exposed surface.

As you add one rod, you experience a *net* gain of 18 square centimeters (i.e., a loss of 6 square centimeters, together with a gain of 24 square centimeters.)

To make a ladder of n rods, you start with one rod (area: 30 cm.^{2}), and then add $(n - 1)$ more (area: $(n - 1) \times 18$). Thus, the total area is:

$$A = 30 + 18 \times (n - 1).$$

Now! Are these "different" formulas really different? This is a fine exercise in the use of the distributive law, etc. Can you make the derivations necessary to show that all these formulas describe the same function?